GEORGIA INSTITUTE OF TECHNOLOGY — OFFICE OF CONTRACT ADMINISTRATION

PROJECT ADMINISTRATION DATA SHEET

[X] ORIGINAL [ ] REVISION NO. ______

Project No. E-21-628 — DATE 5/4/82

Project Director: F. L. Lewis — School/~~Lab~~ Electrical Engineering

Sponsor: National Science Foundation, Washington, D.C. 20550

Type Agreement: Grant No. ECS-8204656

Award Period: From 7/1/82 To ~~12/31/83*~~ 12/31/84 (Performance) ______ (Reports)

Sponsor Amount: $22,191 — Contracted through:

Cost Sharing: $7,394 (E-21-376) — GTRI/~~GIT~~

Title: ~~Research Initiation:~~ Extension of Geometric System Theory of Descriptor Systems

ADMINISTRATIVE DATA

OCA Contact William F. Brown X4820

1) Sponsor Technical Contact:

Abraham H. Haddad, Program Official
Division of Electrical, Computer and
Systems Engineering
National Science Foundation
Washington, D.C. 20550
(202) 357-9618

2) Sponsor Admin/Contractual Matters:

Lee A. DeHerrera, Grants Official
Division of Grants & Contracts
Directorate for Administration
National Science Foundation
Washington, D.C. 20550
(202) 357-9602

Defense Priority Rating: None

Security Classification: None

RESTRICTIONS

See Attached NSF Supplemental Information Sheet for Additional Requirements.

Travel: Foreign travel must have prior approval – Contact OCA in each case. Domestic travel requires sponsor approval where total will exceed greater of $500 or 125% of approved proposal budget category.

Equipment: Title vests with GIT

COMMENTS:

* Includes the usual six-month unfunded flexibility period.

RECEIVED MAY 1982 Research Reports Office

COPIES TO:

Research Administrative Network

~~[illegible]~~	Research Security Services	EES Public Relations (2)
Research Property Management	~~Reports Coordinator (OCA)~~	Computer Input
Accounting	Legal Services (OCA)	Project File
Procurement/EES Supply Services	Library	Other ______

FORM OCA 4:781

GEORGIA INSTITUTE OF TECHNOLOGY | OFFICE OF CONTRACT ADMINISTRATION

SPONSORED PROJECT TERMINATION/CLOSEOUT SHEET

Date 4/2/86

Project No. E-21-628 School/~~XXX~~ EE

Includes Subproject No.(s) N/A

Project Director(s) F. L. Lewis GTRC / ~~XXX~~

Sponsor National Science Foundation

Title ~~Research Initiation~~: Extension of Geometric System Theory of Descriptor Systems

Effective Completion Date: 12/31/84 (Performance) 3/31/85 (Reports)

Grant/Contract Closeout Actions Remaining:

Note to P.I. NSF form 98A has been submitted to sponsor. Two copies of final technical report should be furnished to PPC.

- [XXX] None
- [ ] Final Invoice or Final Fiscal Report
- [ ] Closing Documents
- [ ] Final Report of Inventions - Sent Questionnaire to P.I.
- [ ] Govt. Property Inventory & Related Certificate
- [ ] Classified Material Certificate
- [ ] Other

Continues Project No. Continued by Project No.

COPIES TO:

Project Director
Research Administrative Network
Research Property Management
Accounting
Procurement/EES Supply Services
Research Security Services
Reports Coordinator (OCA)
Legal Services

Library
GTRI
Research Communications (2)
Project File
Other M. Heyser
A. Jones
R. Embry

Form OCA 60:1028

APPENDIX VI

NATIONAL SCIENCE FOUNDATION
Washington, D.C. 20550

FINAL PROJECT REPORT
NSF FORM 98A

PLEASE READ INSTRUCTIONS ON REVERSE BEFORE COMPLETING

PART I—PROJECT IDENTIFICATION INFORMATION

1. Institution and Address	2. NSF Program	3. NSF Award Number
School of Electrical Engineering Georgia Institute of Technology Atlanta, GA 30332	STOR	ECS-8204656
	4. Award Period From 7/1/82 To 12/31/84	5. Cumulative Award Amount $47,237

6. Project Title

Extension of Geometric System Theory to Descriptor Systems

PART II—SUMMARY OF COMPLETED PROJECT *(FOR PUBLIC USE)*

Descriptor systems of the form $E\dot{x} = Ax + Bu$ were studied, where E is singular and $x \in R^n$. The concepts of deflating subspace and the Weierstrass form were first examined in the context of the relative eigenstructure of E and A. It was shown that any decomposition of R^n into two deflating subspaces induces a corresponding decomposition of the system into forward and backward (or slow and fast) subsystems. These ideas were related to Luenberger's double sweep method.

The inversion problem for descriptor systems was solved, and a structure algorithm was presented which generates the admissible initial conditions and other subspaces. Next, the observability, controllability, and pole assignment problems were examined, and the open-loop control and estimation problems were solved in the discrete case. Descriptor reachability and observability matrices were defined and used in this work. Moore's eigenstructure assignment method was generalized to descriptor systems. The singular optimal control problem for singular systems was solved by an approach which depends on stable algorithms to reveal the problem structure and then solution of a Riccati equation.

The Cayley-Hamilton and Bézout theorems and Fadeev's method were extended to descriptors.

Wonham's algorithms to generate reachable subspaces were extended to singular systems in K. Ozcaldiran's Ph.D. work, and the disturbance decoupling problem was investigated.

PART III—TECHNICAL INFORMATION *(FOR PROGRAM MANAGEMENT USES)*

1. ITEM *(Check appropriate blocks)*	NONE	ATTACHED	PREVIOUSLY FURNISHED	TO BE FURNISHED SEPARATELY TO PROGRAM: Check (√)	TO BE FURNISHED SEPARATELY TO PROGRAM: Approx. Date
a. Abstracts of Theses		XX			
b. Publication Citations		XX			
c. Data on Scientific Collaborators		XX			
d. Information on Inventions					
e. Technical Description of Project and Results				XX	5/1/85
f. Other *(specify)*					

2. Principal Investigator/Project Director Name *(Typed)*	3. Principal Investigator/Project Director Signature	4. Date
Frank L. Lewis		3/25/85

NSF Form 98A (5-78) Supersedes All Previous Editions

Form Approved OMB No. 99R0013

DIVISION OF
GRADUATE STUDIES AND RESEARCH

GEORGIA INSTITUTE OF TECHNOLOGY

OCT 19 1984

REQUEST FOR APPROVAL OF THESIS TOPIC

Date August 14, 1984

NAME Kadri Ozcaldiran
First Middle Last

requests approval to prepare and present a thesis in partial fulfillment of the requirements for the degree of Ph. D.

Thesis Title: Control of Descriptor Systems

Brief description:

Linear, time-invariant descriptor systems, that is, systems of the form

$$\left\{ \begin{array}{l} E\dot{x}=Ax+Bu \\ y=Cx+Du \end{array} \right\} \quad \ldots\ldots\ldots (*)$$

are investigated. The eigenstructure analysis of the two operators E and A is carried out and is shown to yield a decomposition of (*) into two subsystems.

A descriptor variable feedback is designed to shift the infinite eigenvalues of (*) to the complex plane and to eliminate the impulsive behavior of the trajectories of (*). Eigenstructure assignment using descriptor variable feedback is also investigated.

The family of reachability subspaces and the reachability indices of the triple (E,A,B), and their relation to the geometric analysis of (*) will also be studied.

APPROVED:

Director of School

Graduate Student requesting approval of thesis topic

F. Lewis

Thesis Advisor

35022

Box No.

10/22/84

Member Reading Committee or Thesis Advisory Committee

Dean, Division of Graduate Studies and Research

Member Reading Committee or Thesis Advisory Committee

Prepare original only - Graduate Division will release copies.

FRANK L. LEWIS

PUBLICATIONS UNDER ECS-8204656

Invited Papers

1. F. L. Lewis and K. Ozcaldiran, "Reachability and controllability for descriptor systems," 27th MSCAS, Morgantown, WV, June 1984.

2. F. L. Lewis, "Descriptor systems: fundamental matrix, reachability and observability matrices, subspaces," Proc. 23rd IEEE Conf. Decision and Control, December 1984.

Journal Articles

1. F. L. Lewis, "Descriptor systems: expanded descriptor equation and Markov parameters," IEEE Trans. Automat. Contr., vol. AC-28, no. 5, pp. 623-627, May 1983.

2. F. L. Lewis, "Descriptor systems: decomposition into forward and backward subsystems," IEEE Trans. Automat. Contr., vol. AC-29, no. 2, pp. 167-170, Feb. 1984.

3. F. L. Lewis, "Fundamental, reachability, and observability matrices for descriptor systems," IEEE Trans. Automat. Contr., May 1985.

Conference Papers with Proceedings

1. F. L. Lewis, "Inversion of descriptor systems," Proc. ACC, San Francisco, CA, pp. 1153-1158, June 1983.

2. F. L. Lewis, "Adjoint matrix, Bezout theorem, Cayley-Hamilton theorem, and Fadeev's method for the matrix pencil (sE-A)," Proc. IEEE Conf. Decision and Control, pp. 1282-1288, Dec. 1983.

3. K. Ozcaldiran and F. L. Lewis, "A result on the placement of infinite eigenvalues in descriptor systems," Proc. ACC, San Diego, CA, June 1984.

Conference Papers without Proceedings

1. F. L. Lewis and K. Ozcaldiran, "The relative eigenstructure problem and descriptor systems," SIAM National Meeting, Denver, CO, June 1983.

Submitted or in Preparation

1. F. L. Lewis and K. Ozcaldiran, "Reachability, controllability, and feedback assignment of eigenstructure for descriptor systems."

2. F. L. Lewis, "Optimal control for singular systems."

3. F. L. Lewis, "On the eigenstructure assignment of singular systems."

4. F. L. Lewis, "Inversion of singular systems."

Ph.D. Theses

1. K. Ozcaldiran, "Control of descriptor systems."

Kadri Özçaldiran

Biography

Kadri Ozcaldiran was born in Karşıyaka-izmir, Turkey. He received the B.S. degree in electrical engineering from Middle East Technical University in 1979 and the M.S.E.E. degree from Georgia Institute of Technology in 1980. He is currently working on his Ph. D. in electrical engineering and his M.S. in mathematics at Georgia Tech. Since 1980 he has been employed as a teaching assistant by the Schools of Electrical Engineering and Mathematics. His interests include control theory, mathematical system theory, and power systems. He is an associate member of Sigma Xi.

Descriptor Systems: Expanded Descriptor Equation and Markov Parameters

FRANK LEWIS

Abstract —Generalized concepts of solvability and conditionability are presented for descriptor systems, and tests are given for these generalized properties. An expression is derived for descriptor system Markov parameters. The notion of "expanded descriptor equation" is presented.

Manuscript received April 12, 1982; revised September 24, 1982. This work was supported by the National Science Foundation under Grant ECS-8204656.
The author is with the School of Electrical Engineering, Georgia Institute of Technology, Atlanta, GA 30332.

0018-9286/83/0500-0623$01.00

I. Introduction

In modeling large-scale systems with the usual state-space formulation, structural information is sometimes lost. In fact, in some cases, a state-space formulation does not even exist [18]. The descriptor system formulation always exists and retains the physically meaningful structure of a large-scale system. It is therefore useful in econometrics, energy system modeling, robotics, decentralized control and decision networks, etc.

We present generalized notions of solvability and conditionability for descriptor systems, along with tests which reduce in the regular case to previously known results. We also derive expressions for descriptor system Markov parameters. An example is presented to show some of the subtleties involved.

II. Background

Consider the following linear discrete-time system over the real numbers $\mathfrak{R}$:

$$Ex_{k+1} = Fx_k + Gu_k \tag{1a}$$

$$y_k = Hx_k + Ju_k; \qquad k = 0, 1, \cdots, N-1 \tag{1b}$$

where k is a nonnegative integer, $u_k \in \mathfrak{R}^m$, $x_k \in \mathfrak{R}^n$, $y_k \in \mathfrak{R}^p$, and E, F, G, H, J, are appropriately dimensioned constant matrices. System (1) will be called a *descriptor system* and the x_k will be called *descriptor variables*. We may write (1) in expanded form as

$$\begin{bmatrix} E & -F & 0 & \cdots & 0 & 0 & 0 \\ 0 & E & -F & \cdots & 0 & 0 & 0 \\ \vdots & & \ddots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & E & -F & 0 \\ 0 & 0 & 0 & \cdots & 0 & E & -F \end{bmatrix} \begin{bmatrix} x_N \\ x_{N-1} \\ \vdots \\ x_1 \\ x_0 \end{bmatrix} = \begin{bmatrix} G & 0 & \cdots & 0 & 0 \\ 0 & G & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & G & 0 \\ 0 & 0 & \cdots & 0 & G \end{bmatrix} \begin{bmatrix} u_{N-1} \\ u_{N-2} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix} \tag{2a}$$

$$\begin{bmatrix} y_{N-1} \\ y_{N-2} \\ \vdots \\ y_1 \\ y_0 \end{bmatrix} = \begin{bmatrix} H & 0 & \cdots & 0 & 0 \\ 0 & H & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & H & 0 \\ 0 & 0 & \cdots & 0 & H \end{bmatrix} \begin{bmatrix} x_{N-1} \\ x_{N-2} \\ \vdots \\ x_1 \\ x_0 \end{bmatrix} + \begin{bmatrix} J & 0 & \cdots & 0 & 0 \\ 0 & J & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & J & 0 \\ 0 & 0 & \cdots & 0 & J \end{bmatrix} \begin{bmatrix} u_{N-1} \\ u_{N-2} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix} \tag{2b}$$

By appropriate definition of the coefficient block matrices A_N, B_N, C_N, D_N and of the input, descriptor, and output sequence vectors $\bar{u}_{0,N}$, $\bar{x}_{0,N}$, $\bar{y}_{0,N}$ we can rewrite (2) as

$$A_N \bar{x}_{0,N+1} = B_N \bar{u}_{0,N} \tag{3a}$$

$$\bar{y}_{0,N} = C_N \bar{x}_{0,N} + D_N \bar{u}_{0,N}. \tag{3b}$$

If $E = I$, the identity matrix, then (1) is just the familiar state-space formulation.

III. Solvability and Conditionability

Existence of Solution

A solution $\bar{x}_{0,N+1}$ to (3a) exists for all $\bar{u}_{0,N}$ if and only if [4], [14]

$$R(A_N) \supset R(B_N) \tag{4}$$

where $R(A_N)$ represents the range of A_N, in which case we say that (1) is *solvable*. Then a solution is given by [4]

$$\bar{x}_{0,N+1} = A_N^- B_N \bar{u}_{0,N} + (I - A_N^- A_N)\bar{z}_{0,N+1} \tag{5}$$

where A_N^- is a generalized inverse of A_N. The vector $\bar{z}_{0,N+1}$ is arbitrary and consists of the additional conditions z_k which must be specified at each time k to give a unique solution. The number of additional conditions required is equal to the rank of deficiency of $(I - A_N^- A_N)$. See the discussion in [5]. We have the following new theorem.

Theorem 1: System (1) is solvable if and only if

$$\operatorname{rank}[zE - F \;:\; G] = \operatorname{rank}[zE - F] \quad \text{a.e.} \tag{6}$$

where equality "a.e." indicates equality except for a finite number of z.

Proof: Define $q = [q_1 \,:\, q_2 \,:\, \cdots \,:\, q_N]^T$ and $p(z) = q_1 + q_2 z^{-1} + \cdots + q_N z^{-(N-1)}$ where $q_i \in \mathfrak{R}^n$. Then

$$q^T A_N = [q_1 E \;:\; (-q_1 F + q_2 E) \;:\; \cdots \;:\; (-q_{N-1}F + q_N E) \;:\; q_N F]$$

and

$$p(z)(zE - F) = q_1 Ez + (-q_1 F + q_2 E) + \cdots + (-q_{N-1}F + q_N E)z^{-(N-2)} - q_N F z^{-(N-1)}.$$

Hence, $q^T A_N = 0$ if and only if $p(z)(zE - F) = 0$ for all z.

Furthermore, $q^T B_N = [q_1 G \,:\, q_2 G \,:\, \cdots \,:\, q_N G]$ and $p(z)G = q_1 G + q_2 G z^{-1} + \cdots + q_N G z^{-(N-1)}$.

Hence, $q^T B_N = 0$ if and only if $p(z)G = 0$ for all z.

Now suppose that $R(A_N) \supset R(B_N)$. Then $q^T A_N = 0 \Rightarrow q^T B_N = 0$. Assume that $p(z)\cdot(zE - F) = 0$ for all z. Then $q^T A_N = 0$, which implies that $q^T B_N = 0$ and hence $p(z)G = 0$ for all z. Therefore, $R(zE - F) \supset R(G)$ except for a finite number of G, which is equivalent to (6).

The proof of sufficiency follows in a similar manner. ■

Note that if $G = I$, then our definition is equivalent to that of Luenberger [1], and condition (6) reduces to his requirement for the regularity of $[zE - F]$ ($[zE - F]$ is *regular* if $\operatorname{rank}[zE - F] = n$ a.e.). See also [2]. We have simply generalized the notion of solvability to the case of arbitrary G, and so we have a sufficient *and* necessary condition for existence of a solution to (1).

Comparing (6) to the reachability condition that $\operatorname{rank}[zE - F \,:\, G] = n$ for all z [3], [7], we are led to speculate upon the existence of a deeper connection between solvability and reachability than has yet been established.

Uniqueness of Solution

Assuming that (3a) is solvable, then solutions $\bar{x}_{0,N+1}$ arising from the same input give rise to the same output (i.e., the solution is unique with respect to $\bar{y}_{0,N+1}$) if and only if

$$N(A_N) \subset N(C_{N+1}) \tag{7}$$

where $N(A_N)$ represents the null space of A_N, for then the arbitrary component of the solution $\bar{x}_{0,N}$ arising from $\bar{z}_{0,N+1}$ in (5) is in the null space of C_{N+1} and does not appear in $\bar{y}_{0,N+1}$. We have the following new theorem.

Theorem 2: Suppose that (3a) is solvable and that $x_0 \in N(E)$ and $x_N \in N(F)$. Then the solution is unique with respect to $\bar{y}_{0,N+1}$ if and only if

$$\operatorname{rank}\begin{bmatrix} zE - F \\ H \end{bmatrix} = \operatorname{rank}[zE - F] \text{ a.e.} \tag{8}$$

Proof: Define $p(z) = x_N + x_{N-1}z + \cdots + x_0 z^N$. Then

$$A_N \bar{x}_{0,N+1} = \begin{bmatrix} Ex_N - Fx_{N-1} \\ Ex_{N-1} - Fx_{N-2} \\ \vdots \\ Ex_1 - Fx_0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}^{\text{ii}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \hline 0 \\ -1 \\ 0 \end{bmatrix} u_0 + \begin{bmatrix} x_1^3 \\ x_1^2 \\ 0 \\ \hline 0 \\ x_1^2 \\ x_0^1 \end{bmatrix},$$

$$\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}^{\text{iii}} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \hline 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix} u_0 + \begin{bmatrix} x_1^3 \\ a \\ 0 \\ \hline 0 \\ a \\ x_0^1 \end{bmatrix} \tag{E8}$$

where, in each case, the second term, which comes from $\mathcal{P}_{N(E-F)}\bar{z}_{0,2}$, is arbitrary and consists of the $n = 3$ boundary conditions required to specify a unique solution.

We can identify $\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}^{\text{i}}$ as a "maximally forward" solution which has the greatest number, $\operatorname{rank}(E) = \rho = 2$, of boundary conditions specified at time $k = 0$. Likewise, $\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}^{\text{ii}}$ corresponds to a "maximally backward" solution having the greatest number, $\operatorname{rank}(A) = 2$, of boundary conditions specified at $k = N = 1$. $\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}^{\text{iii}}$ has additional conditions "evenly split" between the initial and final times, and this symmetry derives from the fact that $(I - A_1^{\text{iii}}A_1)$ is the orthogonal projection onto $N(A_1)$, i.e., $A_1^{\text{iii}} = A_1^+$.

Of course, the first question suggested by this example is whether A_1^{i}, A_1^{ii}, A_1^{iii} always generate maximally forward, maximally backward, and evenly split solutions, respectively, for all N when used to find M_k using (15). Further research should reveal the answer to this question.

VII. Conclusion

Generalized notions of solvability and conditionability for descriptor systems were presented which reduce, in the regular case, to familiar results. We derived expressions for descriptor system Markov parameters and defined the expanded descriptor equation. We also defined the forward and backward expanded descriptor equations, and showed that additional conditions can be specified at intermediate times over the solution interval to result in a unique output for a given input. These additional conditions can be considered as a "fictitious input" to the system. A simple example was presented to show some of the subtleties involved.

References

[1] D. G. Luenberger, "Time-invariant descriptor systems," *Automatica*, vol. 14, pp. 473–480, 1978.

[2] F. R. Gantmacher, *The Theory of Matrices*. New York: Chelsea, 1974.

[3] G. C. Verghese, B. C. Levy, and T. Kailath, "A generalized state-space for singular systems," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 811–831, Aug. 1981.

[4] C. R. Rao and S. K. Mitra, *Generalized Inverse of Matrices and Its Applications*. New York: Wiley, 1971.

[5] D. G. Luenberger, "Dynamic equations in descriptor form," *IEEE Trans. Automat. Contr.*, vol. AC-22, pp. 312–321, June 1977.

[6] R. F. Sincovec, A. M. Erisman, E. L. Yip, and M. A. Epton, "Analysis of descriptor systems using numerical algorithms," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 139–147, Feb. 1981.

[7] E. L. Yip and R. F. Sincovec, "Solvability, controllability, and observability of continuous descriptor systems," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 702–707, June 1981.

[8] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1980.

[9] H. H. Rosenbrock, *State-Space and Multivariable Theory*. New York: Wiley, 1970.

[10] G. C. Verghese and T. Kailath, "Impulsive behavior in dynamical systems: Structure and significance," in *Proc. 4th Int. Symp. Math. Theory, Networks Syst.*, Delft, The Netherlands, July 1979, pp. 162–168.

[11] G. Verghese, P. Van Dooren, and T. Kailath, "Properties of the system matrix of a generalized state-space system," *Int. J. Contr.*, vol. 30, no. 2, pp. 235–243, 1979.

[12] I. I. Hirschman, Ed., *Studies in Real and Complex Analysis*, MAA Studies in Mathematics, vol. 3. Englewood Cliffs, NJ: Prentice-Hall, 1965.

[13] P. Hartman and A. Wintner, "The spectra of Toeplitz's matrices," *Amer. J. Math.*, vol. 76, pp. 867–882, 1954.

[14] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*. New York: Krieger, 1980.

[15] B. Dziurla and R. Newcomb, "The Drazin inverse and semi-state equations," in *Proc. 4th Int. Symp. Math. Theory, Networks Syst.*, Delft, The Netherlands, July 1979, pp. 283–289.

[16] S. L. Campbell, C. D. Meyer, Jr., and N. J. Rose, "Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients," *SIAM J. Appl. Math.*, vol. 31, pp. 411–425, Nov. 1976.

[17] T. B. Cline, R. E. Larson, D. G. Luenberger, D. N. Stengel, and K. D. Wall, "Descriptor variable representation of large-scale deterministic systems," Systems Control, Inc., Palo Alto, CA, Tech. Memo. 5168-1, Aug. 31, 1976.

[18] S. P. Singh and R. W. Liu, "Existence of state equation representation of linear large-scale dynamical systems," *IEEE Trans. Circuit Theory*, vol. CT-20, pp. 239–246, May 1973.

0018-9286/83/0500-0627$01.00 ©1983 IEEE

to the input-dependent portion of the ESE:

$$\bar{y}_{0,N} = \begin{bmatrix} J & HG & HFG & \cdots & HF^{N-2}G \\ 0 & J & HG & \cdots & HF^{N-3}G \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & HG \\ 0 & 0 & 0 & \cdots & J \end{bmatrix} \bar{u}_{0,N}.$$

If $x_0 \notin N(E)$, $x_N \notin N(F)$, then we must add to (9) a term describing boundary condition effects (i.e., the zero-input portion of the EDE).

An analogous development holds if $[zE - F]$ is not regular, but (6) holds.

V. The Forward EDE

In some applications, it might be of interest to know which states can be reached by propagating system (1) forward in time from given initial conditions x_0. To investigate this, we need the forward propagation operators $\alpha = E^- F$ and $\beta = E^- G$ where E^- is a generalized inverse of E. Then a solution to (1a), if it exists, is given by [4]

$$\begin{aligned} x_{k+1} &= E^- Fx_k + E^- Gu_k + (I - E^- E) z_k \\ &= \alpha x_k + \beta u_k + \gamma z_k \end{aligned} \tag{17}$$

where z_k is arbitrary and we have defined $\gamma = (I - E^- E)$, the projection onto $N(E)$. We shall not discuss here conditions under which the solution given by (17) exists.

By iteration, we can write the *forward EDE* for (1) as

$$\begin{bmatrix} x_N \\ \hline y_{N-1} \\ y_{N-2} \\ \vdots \\ y_1 \\ y_0 \end{bmatrix} = \begin{bmatrix} \alpha^N \\ \hline H\alpha^{N-1} \\ H\alpha^{N-2} \\ \vdots \\ H\alpha \\ H \end{bmatrix} x_0 + \begin{bmatrix} \beta & \alpha\beta & \alpha^2\beta & \cdots & \alpha^{N-1}\beta \\ \hline J & H\beta & H\alpha\beta & \cdots & H\alpha^{N-2}\beta \\ 0 & J & H\beta & \cdots & H\alpha^{N-3}\beta \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & J & H\beta \\ 0 & 0 & \cdots & 0 & J \end{bmatrix} \begin{bmatrix} u_{N-1} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix} + \begin{bmatrix} \gamma & \alpha\gamma & \alpha^2\gamma & \cdots & \alpha^{N-1}\gamma \\ \hline 0 & H\gamma & H\alpha\gamma & \cdots & H\alpha^{N-2}\gamma \\ 0 & 0 & H\gamma & \cdots & H\alpha^{N-3}\gamma \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & & H\gamma \\ 0 & 0 & \cdots & & 0 \end{bmatrix} \begin{bmatrix} z_{N-1} \\ \vdots \\ z_1 \\ z_0 \end{bmatrix}. \tag{18}$$

This is of the same form as the expanded state equation (ESE) [8]. Note that the vector of z_k's is just the vector of additional conditions which could be specified at each time to make the solution unique. Compare this to the discussion in [5]. We see that these additional conditions specified at intermediate times during the interval $[0, N]$ have the same effect as an additional input.

We could similarly define the *backward EDE* based on the backward propagation operators $F^- E$, $F^- G$, $(I - F^- F)$.

VI. Example

The following simple example reveals some of the effects of different choices for $[E \quad -F]^- = \begin{bmatrix} A \\ -B \end{bmatrix}$.

Let (1a) be

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_{k+1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_k \tag{E1}$$

and $N = 1$. Then (2a) becomes

$$\begin{aligned} A_1 \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} &= [E \quad -F] \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_0. \end{aligned} \tag{E2}$$

Clearly, $(zE - F)$ is regular; so a solution is given by (5):

$$\begin{bmatrix} x_1 \\ x_0 \end{bmatrix} \triangleq \begin{bmatrix} x_1^3 \\ x_1^2 \\ x_1^1 \\ \hline x_0^3 \\ x_0^2 \\ x_0^1 \end{bmatrix} = A_1^- Gu_0 + (I - A_1^- A_1)\bar{z}_{0,2} \tag{E3}$$

for any w. The second term can be written as

$$(I - A_1^- A_1)\bar{z}_{0,2} = \mathcal{P}_{N(A_1)}\bar{z}_{0,2} \tag{E4}$$

where $\mathcal{P}_{\mathcal{X}}$ represents a (not necessarily orthogonal) projection onto subspace $\mathcal{X}$.

Now consider the following three possibilities for $\begin{bmatrix} A \\ -B \end{bmatrix} = A_1^-$ [14]:

$$\text{i)} \quad A_1^{\text{i}} \triangleq \begin{bmatrix} E^+ \\ 0 \end{bmatrix} - \begin{bmatrix} E^+ \\ F^+ \end{bmatrix} \left(\mathcal{P}^0_{R^\perp(E)} + \mathcal{P}^0_{R^\perp(F)} \right)^+ \mathcal{P}^0_{R^\perp(E)}$$

$$\text{ii)} \quad A_1^{\text{ii}} \triangleq -\begin{bmatrix} 0 \\ F^+ \end{bmatrix} + \begin{bmatrix} E^+ \\ F^+ \end{bmatrix} \left(\mathcal{P}^0_{R^\perp(E)} + \mathcal{P}^0_{R^\perp(F)} \right)^+ \mathcal{P}^0_{R^\perp(F)} \tag{E5}$$

$$\text{iii)} \quad A_1^{\text{iii}} \triangleq \begin{bmatrix} E^+ \\ -F^+ \end{bmatrix} (EE^+ + FF^+)^+ \tag{E5}$$

where $\mathcal{P}^0_{\mathcal{X}}$ represents the orthogonal projection onto subspace $\mathcal{X}$ and superscript "+" represents the Moore–Penrose matrix inverse [4]. In this example, $E = E^+$, $F = F^+$, and

$$\mathcal{P}^0_{R^\perp(E)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{P}^0_{R^\perp(F)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{E6}$$

Therefore,

$$A_1^{\text{i}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1^{\text{ii}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1^{\text{iii}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{E7}$$

Corresponding to these three possible choices for A_1^-, we have three different possible solutions (E3) given by

$$\begin{bmatrix} x_1 \\ x_0 \end{bmatrix}^{\text{i}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{bmatrix} u_0 + \begin{bmatrix} x_1^3 \\ x_0^2 \\ 0 \\ \hline 0 \\ x_0^2 \\ x_0^1 \end{bmatrix},$$

and

$$(zE-F)p(z) = -Fx_N + (Ex_N - Fx_{N-1})z + \cdots + (Ex_1 - Fx_0)z^N + Ex_0 z^{N+1}.$$

Hence, if $x_0 \in N(E)$, $x_N \in N(F)$, then $A_N \bar{x}_{0,N+1} = 0$ and if and only if $(zE-F)p(z) = 0$ for all z. Furthermore,

$$C_{N+1}\bar{x}_{0,N+1} = \begin{bmatrix} Hx_N \\ Hx_{N-1} \\ \vdots \\ Hx_0 \end{bmatrix}$$

and $Hp(z) = Hx_N + Hx_{N-1}z + \cdots + Hx_0 z^N$. Hence, $C_{N+1}\bar{x}_{0,N+1} = 0$ if and only if $Hp(z) = 0$ for all z.

The remainder of the proof follows an argument such as the one used to prove Theorem 1. ∎

We say that (1) is *conditionable* if there exists a unique $\bar{y}_{0,N+1}$ for all $\bar{x}_{0,N+1}$ satisfying $x_0 \in N(E)$ and $x_N \in N(F)$. Thus, for a conditionable system, any arbitrariness in $\bar{y}_{0,N}$ for a given input $\bar{u}_{0,N}$ arises from variations in x_0 which occur in $N^{\perp}(E)$ or from variations in x_N which occur in $N^{\perp}(F)$. This means that for a conditionable system, the output $\bar{y}_{0,N}$ is uniquely determined by specifying the input $\bar{u}_{0,N}$ and the initial and final states x_0 and x_N.

Note that if $H = I$, then our definition is equivalent to that of Luenberger [1], and condition (8) again reduces to the requirement for regularity of $(zE-F)$.

From Theorem 2, we see that for a conditionable system, any $x_0 \in N(E)$ and $x_N \in N(F)$ will give the same $\bar{y}_{0,N}$ as $x_0 = 0$ and $x_N = 0$. Hence, we can define the *subspace of effective initial conditions* $N^{\perp}(E)$ as the set of x_0 which can make $\bar{y}_{0,N}$ vary from the output sequence which results when $x_0 = 0$. Similarly, we define $N^{\perp}(F)$ as the *subspace of effective final conditions.*

Comparing (8) to the observability condition that rank $\begin{bmatrix} zE-F \\ H \end{bmatrix} = n$ for all z [3], [7], we speculate that a connection exists between conditionability and observability which has not yet been investigated.

IV. Markov Parameters

If (1) is solvable and conditionable, we can write

$$\bar{y}_{0,N} = ([0 \quad C_N]A_N^- B_N + D_N)\bar{u}_{0,N} \tag{9}$$

as the unique solution given $\bar{u}_{0,N}$ and $x_0 \in N(E)$, $x_N \in N(F)$.

The coefficient of $\bar{u}_{0,N}$ in (9) is the matrix of Markov parameters of (1), and (9) will be called the *expanded descriptor equation* (EDE). We shall show later that if $E = I$, then (9) reduces to the input-dependent portion of the expanded state equation (ESE) [8].

The inverse A_N^- is not unique. The next result shows one method of computing A_N^- which uses a right inverse for $[E \quad -F]$. For simplicity, we assume that $(zE-F)$ is regular, i.e., rank $(zE-F) = n$ a.e. [2].

Theorem 3: Let $\begin{bmatrix} A \\ -B \end{bmatrix}$ be any right inverse for $[E \quad -F]$ so that $[E \quad -F]\begin{bmatrix} A \\ -B \end{bmatrix} = I$. Define

$$b = EB, \qquad f = FA \tag{10}$$

and solve for the sequence $\pi_i \in \mathfrak{R}^n$ using the Toeplitz system of equations

$$\begin{bmatrix} -b & I & -f & 0 & \cdots & 0 \\ 0 & -b & I & -f & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & & -b & I & -f \end{bmatrix} \begin{bmatrix} \pi_N \\ \pi_{N-1} \\ \vdots \\ \pi_0 \\ \vdots \\ \pi_{-(N-1)} \\ \pi_{-N} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{11}$$

where I is in the Nth block in the vector on the right-hand side (i.e., in the central block).

Then a generalized inverse for A_N is given by

$$A_N^- = \begin{bmatrix} A\pi_0 - B\pi_1 & A\pi_1 - B\pi_2 & \cdots & A\pi_{(N-1)} - B\pi_N \\ A\pi_{-1} - B\pi_0 & A\pi_0 - B\pi_1 & & \\ A\pi_{-2} - B\pi_{-1} & A\pi_{-1} - B\pi_0 & & \\ \vdots & \vdots & \ddots & \\ A\pi_{-N} - B\pi_{-N+1} & A\pi_{-N+1} - B\pi_{-N+2} & \cdots & A\pi_{-1} - B\pi_0 \end{bmatrix}. \tag{12}$$

Proof: Note that the π_i satisfy the recursion

$$\pi_i = f\pi_{i-1} + b\pi_{i+1} + \delta_{i0}I; \qquad i = 0, \pm 1, \cdots \tag{13}$$

where δ_{i0} is the Kronecker delta. Now it is easily verified that $A_N A_N^- = I$. ∎

We can define the descriptor system state transition matrix as

$$\phi_k = A\pi_k - B\pi_{k+1}; \qquad k = 0, \pm 1, \cdots \tag{14}$$

and the descriptor system Markov parameter as

$$M_k = H\phi_{k-1}G + \delta_{k0}J. \tag{15}$$

Clearly, ϕ_k and M_k will have different properties, depending on the choice of $[E \quad -F]^- = \begin{bmatrix} A \\ -B \end{bmatrix}$. In Section VI, we will see the effects of some different choices of A and B.

Now we can write the EDE (9) as

$$\begin{bmatrix} y_{N-1} \\ y_{N-2} \\ \vdots \\ y_0 \end{bmatrix} = \begin{bmatrix} M_0 & M_1 & M_2 & M_3 & \cdots & M_{N-1} \\ M_{-1} & M_0 & M_1 & M_2 & & M_{N-2} \\ M_{-2} & M_{-1} & M_0 & M_1 & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \\ & & & & & M_1 \\ M_{-(N-1)} & M_{-(N-2)} & \cdots & & & M_0 \end{bmatrix} \begin{bmatrix} u_{N-1} \\ u_{N-2} \\ \vdots \\ u_0 \end{bmatrix}. \tag{16}$$

Notice that M_k for $k > 0$ describes a propagation of input effect forward through time, while M_k for $k < 0$ describes a propagation of input effect backward through time.

To show that the state equation case $(E = I)$ follows from the above, note that if $E = I$, then one choice for $\begin{bmatrix} A \\ -B \end{bmatrix}$ is $\begin{bmatrix} I \\ 0 \end{bmatrix}$. Then $b = 0$, $f = F$, $\pi_k = 0$ for $k < 0$, and $\pi_k = F^k$ for $k \geq 0$. Then

$$M_k = \begin{cases} 0, & k < 0 \\ J, & k = 0 \\ HF^{k-1}G, & k > 0 \end{cases}$$

which are the state equation Markov parameters. In this case, (16) reduces

1983 Automatic Control Conference
San Francisco, CA

FP2 - 2:30

INVERSION OF DESCRIPTOR SYSTEMS

Frank L. Lewis

School of Electrical Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332

Abstract

A modified form of the Luenberger shuffle algorithm and the Silverman structure algorithm are combined to form a descriptor system structure algorithm. This algorithm is used to extend known results on inversion of state-space systems to the case of descriptor systems.

I. INTRODUCTION

The problem of inversion of state-space systems has been solved and is well understood [7,8,10,13-16,18-25]. Silverman's structure algorithm [7,8,15] provides a method for inversion of such systems which yields considerable insight into the properties of the system.

Descriptor, or generalized state space systems have received attention recently due to the fact that they arise naturally in circuit analysis, economics, in the analysis of large scale systems, etc. [1,2,6,26]. Several methods for solving these systems are available [1,3,6,9,27]. Among these is Luenberger's shuffle algorithm [6] which incorporates a test for solvability of a descriptor system.

This paper combines a modified shuffle algorithm with the structure algorithm to form a descriptor system structure algorithm, which is then used to extend known results on state-space system inversion to descriptor systems.

2. BACKGROUND

Consider the descriptor system

$$\underline{S} \qquad Ex_{k+1} = Fx_k + Gu_k \text{, initial condition } x_o \quad (2.1a)$$

$$y_k = Hx_k + Ju_k \text{;} \quad k = 0,1,\ldots,N-1 \quad (2.1b)$$

where E can be singular and $x_k \in R^n$, $u_k \in R^m$, $y_k \in R^p$. If E is singular then x_k should not be considered the state of S, and following Luenberger [1] we call x_k the descriptor variable. Note that if E is singular it is not possible to solve S for x_k by simple iteration given x_o and the input sequence u_k.

We assume that system S is regular, i.e. $|sE-F|$ does not vanish identically [1,3,6,9]. Luenberger has shown [1] that this guarantees existence of solution sequences x_k for all input sequences u_k, and that such solutions can be uniquely specified by fixing boundary conditions for x_k at times k=0 and N.

For any matrix A there exist unitary transformations U and V such that

$$U^*A = \begin{bmatrix} A_r \\ 0 \end{bmatrix} \text{, } AV = \begin{bmatrix} A_c & 0 \end{bmatrix} \qquad (2.2)$$

where superscript "*" denotes complex conjugate transpose and A_r, A_c have full rank. Following Van Dooren [4] we call such transformations row and column compressions, respectively, of A. One method of finding these transformations is provided by the singular value decomposition (SVD) [5] of A

$$A = U \Sigma V^* \text{,} \qquad (2.3)$$

where Σ is diagonal with rank(A) nonzero entries in decreasing order of magnitude. The matrices U and V of (2.3) satisfy (2.2) [4].

We now introduce a modified row compression of an m x n matrix A. In this case, the problem is to find a unitary T such that, given A and a subspace T, then

$$T^*A = \begin{bmatrix} \bar{A} \\ \breve{A} \end{bmatrix} \qquad (2.4)$$

where $\breve{A}$ is a matrix with the maximum number of rows such that $N(\breve{A}) \supset T$ (or equivalently $N^{\perp}[\breve{A}] \subset T^{\perp}$). For our application $T = N(E)$ for a given matrix E, and (2.4) will allow us to write $\breve{A} = ME$ for some M.

For one method of finding T, let P_T denote the orthogonal projection onto T, and $T^{\perp}$ denote the orthogonal complement of T. Then we can write A $= AP_{T^{\perp}} + AP_T \equiv A_1 + A_2$ where $N^{\perp}(A_1) = P_{T^{\perp}} N^{\perp}(A) \subset T^{\perp}$ and $N^{\perp}(A_2) = P_T N^{\perp}(A) \subset T$. Let the SVD of $A_2 = AP_T$ be $A_2 = U \Sigma V^*$. Then

$$U^*A = U^*A_1 + U^*A_2 = \begin{bmatrix} \bar{A}_1 \\ \breve{A}_1 \end{bmatrix} + \begin{bmatrix} \bar{A}_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{A}_1 + \bar{A}_2 \\ \breve{A}_1 \end{bmatrix}$$

where $\bar{A}_2$ has full rank and $N^{\perp}(\breve{A}_1) \subset N^{\perp}(A_1) \subset T^{\perp}$. By construction $\breve{A}_1$ has the maximum number of rows. Hence T=U satisfies (2.4).

Note that if $T = R^n$ then (2.4) reduces to the usual row compression in (2.2). At the other extreme, if $T=0$ then T=I and $\breve{A} = A$.

3. DESCRIPTOR SYSTEM STRUCTURE ALGORITHM

The descriptor system structure algorithm (DSSA) presented herein is a marriage between the Luenberger shuffle algorithm [6] and the Silverman structure algorithm [7,8]. A nice duality emerges here, since the shuffle algorithm advances in time

Research supported by the National Science Foundation under Grant ECS-8204656.

components of the input vector, and the structure algorithm advances in time components of the output vector. A few initial remarks will clarify the presentation.

In the form presented in [6] the shuffle algorithm yields an expression for the descriptor variable of the form

$$x_{k+1} = Ax_k + B_1u_k + B_2u_{k+1} + \cdots , \qquad (3.1)$$

where $u_k, u_{k+1}, \ldots \in R^n$. To make this algorithm suitable for inclusion in a system inversion scheme, it must be modified to yield an expression of the form

$$x_{k+1} = Ax_k + B\hat{u}_k , \qquad (3.2)$$

where $\hat{u}_k \in R^m$. This is accomplished by using input space transformations (step 2a of DSSA) and row operations (step 3 of DSSA). The vector $\hat{u}_k$ will be composed of linear combinations of time-shifted components of u_k.

In order to interface the modified shuffle algorithm with the structure algorithm we will apply the modified row compression (2.4) to the lower portion of the transformed H matrix at each iteration (step 2b of DSSA).

DSSA requires no transformations on the descriptor variable x_k. Vectors u_k and y_k are transformed into related vectors $\hat{u}_k$ and $\hat{y}_k$ respectively. The result of DSSA is a system S^s (3.3) with input $\hat{u}_k$, output $\hat{y}_k$, and within which x_k is a state variable. Thus DSSA always provides a means of solving the regular descriptor system S, since S^s can be solved iteratively to obtain x_k. This is just a modification of the method of solution presented in [6], more efficient in the same sense in which (3.2) is more efficient than (3.1), depending as it does on only one m-vector input rather than on several vectors each with m components.

In case S is invertible, then state system S^s can be immediately inverted to find S^{-1} (4.1), a system which yields sequence $\hat{u}_k$ given x_o and sequence $\hat{y}_k$.

In the initialization of DSSA we use notation like $\breve{A} \in R^{oxn}$. This means that $\breve{A}$ is a "null matrix" with zero rows and n columns.

We now present the algorithm.

__Descriptor System Structure Algorithm (DSSA)__

1. __Initialization:__
   a. system – $E_o = E$, $F_o = F$, $G_o = G$, $H_o = H$, $J_o = J$
   b. transformations – $Y^o = I_p$, $U^o = I_m$ (i.e. identity matrices)
   c. input and output vectors – $u_k^o = u_k$, $y_k^o = y_k$
   d. additional condition information – $\breve{F} \in R^{oxn}$, $\breve{G} \in R^{oxo}$, $\breve{H} \in R^{oxn}$, $\breve{u}_k \in R^{ox1}$, $\breve{y}_k \in R^{ox1}$
   e. i=0.
2. __Iteration i+1:__

$$\left|\begin{array}{c|c} E_i & 0 \\ \hline 0 & I \end{array}\right| \left|\begin{array}{c} x_{k+1} \\ y_k^i \end{array}\right| = \left|\begin{array}{c|c} F_i & G_i \\ \hline H_i & J_i \end{array}\right| \left|\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right| \left|\begin{array}{c} x_k \\ u_k^i \end{array}\right|$$

__step 1__

a. row compression on E_i
b. row compression on J_i

$$\rho_i\left\{\left|\begin{array}{c|c} \bar{E}_i & 0 \\ 0 & \\ \hline 0 & Y_1 \end{array}\right|\right. \qquad \left|\begin{array}{c|c} \bar{F}_i & \bar{G}_i \\ \breve{F}_i & \breve{G}_i \\ \hline \bar{H}_i & \bar{J}_i \\ \tilde{H}_i & 0 \end{array}\right|\}\sigma_i \qquad \left|\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right|$$

__step 2__ (see note 1)

a. column compression on $\breve{G}_i$.
b. modified row compression on $\breve{H}_i$ such that $N(\breve{H}_{i1}) \supset N(\bar{E}_i)$, i.e. $\breve{H}_{i1} = M_i\bar{E}_i$ for some M_i

$$\left|\begin{array}{c|c} \bar{E}_i & 0 \\ 0 & \\ \hline 0 & Y_2Y_1 \end{array}\right| \qquad \begin{array}{c} \\ \\ \\ \\ \pi_i\{ \end{array}\left|\begin{array}{c|cc} \bar{F}_i & \bar{G}_{i2} & \overbrace{\bar{G}_{i1}}^{\mu_i} \\ \breve{F}_i & 0 & \tilde{G}_{i1} \\ \hline \bar{H}_i & \bar{J}_{i2} & \bar{J}_{i1} \\ \breve{H}_{i2} & 0 & 0 \\ \breve{H}_{i1} & 0 & 0 \end{array}\right| \qquad \left|\begin{array}{c|c} I & 0 \\ \hline 0 & U_1^{-1} \end{array}\right|$$

__step 3__ (see note 1) row operations to eliminate G_{i1}, $\bar{J}_{i1}$

$$\left|\begin{array}{c|c} \bar{E}_i & 0 \\ 0 & \\ \hline 0 & Y_2Y_1 \end{array}\right| \qquad \begin{array}{c} \\ \\ \\ \\ \pi_i\{ \end{array}\left|\begin{array}{c|cc} \bar{F}' & \bar{G}_{i2} & \overbrace{0}^{\mu_i} \\ \breve{F}_i & 0 & \tilde{G}_{i1} \\ \hline \bar{H}_i & \bar{J}_{i2} & 0 \\ \breve{H}_{i2} & 0 & 0 \\ \breve{H}_{i1} & 0 & 0 \end{array}\right| \qquad \left|\begin{array}{c|c} I & 0 \\ \hline 0 & U_1^{-1} \end{array}\right|$$

__step 4__

a. shuffle $\tilde{F}_i$ and advance last μ_i components of input vector
b. advance last π_i components of output vector

Define:

$$Y_3 = \left|\begin{array}{cc} I & 0 \\ 0 & zI_{\pi_i} \end{array}\right| , \quad U_2 = \left|\begin{array}{cc} I & 0 \\ 0 & zI_{\mu_i} \end{array}\right|$$

$$\left|\begin{array}{c|c} \bar{E}_i & 0 \\ -\breve{F}_i & \\ \hline 0 & Y_3Y_2Y_1 \end{array}\right| \quad \left|\begin{array}{c|cc} \bar{F}' & \bar{G}_{i2} & 0 \\ 0 & 0 & \tilde{G}_{i1} \\ \hline \bar{H}_1' & \bar{J}i_2 & 0 \\ \breve{H}_{i2} & 0 & 0 \\ & M_i\bar{G}_{i2} & 0 \end{array}\right| \quad \left|\begin{array}{c|c} I & 0 \\ \hline 0 & (U_2U_1)^{-1} \end{array}\right|$$

3. __Update:__
   a. Define composite transformations and rename blocks

$$\left|\begin{array}{c|c} E_{i+1} & 0 \\ \hline 0 & Y_c^{i+1} \end{array}\right| \quad \left|\begin{array}{c|c} F_{i+1} & G_{i+1} \\ \hline H_{i+1} & J_{i+1} \end{array}\right| \quad \left|\begin{array}{c|c} I & 0 \\ \hline 0 & (U_c^{i+1})^{-1} \end{array}\right|$$

b. transformations - $Y^{i+1} = Y_c^{i+1} Y^i$, $U^{i+1} = U_c^{i+1} U^i$

c. input and output vectors -

$$u_k^{i+1} = U_c^{i+1} u_k^i \quad (= U^{i+1} u_k)$$

$$y_k^{i+1} = Y_c^{i+1} y_k^i \quad (= Y^{i+1} y_k)$$

d. additional condition information -

$$\breve{F} = \begin{vmatrix} \breve{F} \\ \breve{F}_i \end{vmatrix} \} \, n-\rho_i$$

<u>if $\mu_i \neq 0$:</u> Define

$$U_1 u_k^i = \begin{matrix} \\ \mu_i \{ \end{matrix} \begin{vmatrix} \bar{u}_k^i \\ \tilde{u}_k^i \end{vmatrix}$$

$$\breve{G} = \begin{vmatrix} \breve{G} & 0 \\ 0 & \underbrace{\breve{G}_{i1}}_{\mu_i} \end{vmatrix} \} \, n-\rho_i \, , \quad \breve{u}_k = \begin{vmatrix} \breve{u}_k \\ \tilde{u}_k^i \end{vmatrix}$$

<u>If $\mu_i = 0$:</u>

$$\breve{G} = \begin{vmatrix} \breve{G} \\ 0 \end{vmatrix} \} \, n-\rho_i$$

<u>If $(p - \sigma_i) \neq 0$:</u> Define

$$Y_1 \, y_k^i = \begin{vmatrix} \bar{y}_k^i \\ \tilde{y}_k^i \end{vmatrix} \} \, p-\sigma_i$$

$$\breve{H} = \begin{vmatrix} \breve{H} \\ \breve{H}_i \end{vmatrix} \} \, p-\sigma_i \, , \quad \breve{y}_k = \begin{vmatrix} \breve{y}_k \\ \tilde{y}_k^i \end{vmatrix}$$

e. If $i = n-1$, go to 4.
If $i < n-1$, set $i = i+1$ and go to 2.

4. <u>Stop</u>: DSSA complete.

a. Define: $\hat{u}_k = u_k^n$, $\hat{y}_k = y_k^n$

α = highest power of z in U^n

β = highest power of z in Y^n

b. State system is

$$\underline{S}^s \qquad x_{k+1} = E_n^{-1} F_n x_k + E_n^{-1} G_n \hat{u}_k \, , \, x_o \qquad (3.3a)$$

$$\hat{y}_k = H_n x_k + J_n \hat{u}_k \qquad (3.3b)$$

END OF ALGORITHM.

<u>Note 1</u>: If $\tilde{G}_i$ has zero rows omit steps 2a, 3. Set $\mu_i = 0$. If $\tilde{H}_i$ has zero rows omit step 2b. Set $\pi_i = 0$.

Some of the termination properties of DSSA are now given.

<u>Theorem 3.1</u>

If S is regular then DSSA results in a system of the form (3.3).

<u>proof</u>: The proof is straightforward and follows from the discussion in [6]. Since $\bar{E}_i$ eventually attains full rank for $i \le n$, then clearly for some $J,K \le n$ we must have zero rows in $\tilde{G}_j$ for $j > J$ and in $\tilde{H}_k$ for $k > K$. ●

Note that, like the shuffle algorithm [6] DSSA contains a test for regularity of S since if on any iteration $\tilde{F}_i$ is a non-null zero matrix then S is not regular.

<u>Corollary 3.2</u>

If S is regular then $\alpha \le n$ and $\beta \le n$.

<u>proof</u>: Immediate by Theorem 3.1. ●

Notice that $\hat{u}_k \in R^m$ contains linear combinations of components of u_k through $u_{k+\alpha}$. Likewise $\hat{y}_k \in R^p$ contains linear combinations of components of y_k through $y_{k+\beta}$. Define

$$\bar{u}_{o,k} = \begin{vmatrix} u_{k-1} \\ \vdots \\ u_1 \\ u_o \end{vmatrix} , \quad \bar{y}_{o,k} = \begin{vmatrix} y_{k-1} \\ \vdots \\ y_1 \\ y_o \end{vmatrix} ,$$

and the corresponding quantities $\bar{\breve{u}}_{o,k}$, $\bar{\breve{y}}_{o,k}$.

Each time step 4a of 2 was performed some information was lost relating components of u_k to initial condition x_o [6]. The quantities updated in 3.d. allow us to express these additional conditions as

$$\breve{F} x_o = -\breve{G} \breve{u}_o \, . \qquad (3.4)$$

Using (3.4) we can define the subspace of admissible initial conditions as

$$X_o = \breve{F}^{-1} R(\breve{G}) \qquad (3.5)$$

where superscript "-1" represents here the inverse image of operator $\breve{F}$ [17] and $R(\breve{G})$ is the range of operator $\breve{G}$. X_o corresponds to the subspace defined in [27] and to the (E and F,G)-invariant subspace of [32]. See also [33]. If G=0, then $X_o = N(F)$ is the subspace H_I defined in [31].

If we know the input $\bar{u}_{o,\alpha+1}$ then we know $\breve{u}_o$ and we can define the subspace of admissible initial conditions given a particular input sequence $\bar{u}_{o,\alpha+1}$ as

$$X_{o/u} = -\breve{F}^{+} \breve{G} \, \breve{u}_o + N(\breve{F}) \qquad (3.6)$$

where superscript "+" represents Moore-Penrose inverse and $N(\breve{F})$ is the null space of $\breve{F}$.

At each iteration certain relations must also hold between components of y_k and initial condition x_o [15]. The quantities updated in 3.d. allow us to express these additional conditions as

$$\breve{y}_o = \breve{H} x_o \, . \qquad (3.7)$$

In the next section we discuss inversion of S. If it is desired to use DSSA only to solve S given x_o and $\bar{u}_{o,k}$, then the algorithm can be applied only to (2.1a) to obtain (3.3a), which can be solved iteratively for x_k. Then (2.1b) gives y_k.

The example in section 5 illustrates the application of DSSA.

4. DESCRIPTOR SYSTEM INVERSION

The results of this section are well known for state systems. We merely use DSSA to extend them to regular descriptor systems.

A least-square inverse for S^s is given by [7,10,15]

$$\underline{S}^{-1} \quad x_{k+1} = E_n^{-1}(F_n - G_n J_n^+ H_n)x_k + E_n^{-1} G_n J_n^+ \hat{y}_k + E_n^{-1}(I - J_n^+ J_n)v_k$$

$$\hat{u}_k = -J_n^+ H_n x_k + J_n^+ \hat{y}_k + (I - J_n^+ J_n)v_k \qquad (4.1)$$

for any v_k. We now define two types of descriptor system invertibility.

We say S is left invertible with delay δ if for every $k > 0$, $\bar{u}_{o,k}$ can be uniquely determined by $\bar{y}_{o,k+\delta}$ when $x_o = 0$ [7,13,15,17]. We say S is right invertible with delay δ if for every $k > 0$ and $\bar{y} \in R^{pk}$ there exists a $\bar{u}_{o,k+\delta}$ such that $\bar{y}_{\delta,k+\delta} = \bar{y}$ when $x_o = 0$ [14,15,17,20]. Then we have the following results.

Theorem 4.1

System S is left invertible (with delay β) if and only if J_n has full column rank m. Then $\hat{\bar{u}}_{o,k}$ provided by S^{-1} and the additional conditions (3.4) provide the information required to uniquely reconstruct sequence $\bar{u}_{o,k}$ given $\bar{y}_{o,k+\beta}$.

proof: Sufficiency.

Sequence $\hat{\bar{y}}_{o,k}$ is uniquely determined by the given $\bar{y}_{o,k+\beta}$. Then if J_n has full column rank m, S^{-1} provides a unique value for $\hat{\bar{U}}_{o,k}$ [10,15]. From $\hat{\bar{U}}_{o,k}$ can be reconstructed the sequence $\bar{u}_{o,k}$ except for the components of $\bar{u}_{o,k}$ about which information was lost due to the time-shifts of step 4a [6]. Now, $\breve{G}$ has full column rank by construction and x_o is assumed to be admissible (in fact, in the definition of left invertibility $x_o = 0$). Therefore, (3.4) can be solved uniquely for u_o, which provides the additional information required to solve for the "missing" components of $\bar{u}_{o,k}$.

Necessity

This portion of the proof is the discrete-time analog of the proof given in [7] except for a minor modification to take into account the nonsingular transformation Y_2. ●

Theorem 4.2

System S is right invertible (with delay β) if and only if J_n has full row rank p. Then the desired $\bar{y} = \bar{y}_{\beta,k+\beta}$ and the additional conditions (3.7) provide the information required to construct $\hat{\bar{y}}_{o,k+\beta}$. Sequence $\hat{\bar{u}}_{o,k+\beta}$ provided by S^{-1} and the additional conditions (3.4) then give the information needed to construct the required input $\bar{u}_{o,k+\beta}$.

proof: Sufficiency.

Given $\bar{y} = \bar{y}_{\beta,k+\beta} \in R^{pk}$ and the additional conditions (3.7) we can find $\hat{\bar{y}}_{o,k+\beta}$. Then if J_n has full row rank p, S^{-1} yields a $\hat{\bar{u}}_{o,k+\beta}$ which when applied to S^s gives output $\hat{\bar{y}}_{o,k+\beta}$ [15]. In turn, $\hat{\bar{u}}_{o,k+\beta}$ and additional conditions (3.4) allow the calculation of $\bar{u}_{o,k+\beta}$, the required input to S to produce the desired $\bar{y}_{\beta,k+\beta}$.

Necessity

See [15]. ●

Note that although our definitions of invertibility specify $x_o = 0$, our method allows us to solve the left and right system inversion problems given any admissible x_o.

5. EXAMPLE

$$\underline{S} \quad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} x_{k+1} = \begin{vmatrix} 0 & -2 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & -3 \end{vmatrix} x_k + \begin{vmatrix} 0 & 2 \\ 0 & 0 \\ 1 & 0 \end{vmatrix} u_k, x_o$$

$$y_k = \begin{vmatrix} 1 & 3 & 0 \\ 1 & 0 & -1 \end{vmatrix} x_k.$$

1. Initialize

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & -2 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & -3 \end{vmatrix} \quad \begin{vmatrix} 0 & 2 \\ 0 & 0 \\ 1 & 0 \end{vmatrix}$$

$$Y^o = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad \begin{vmatrix} 1 & 3 & 0 \\ 1 & 0 & -1 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \quad U^o = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

2. Iteration 1: (i=0), step 1a, 1b, 2a, 2b

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & -2 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & -3 \end{vmatrix} \quad \begin{vmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{vmatrix}$$

$$Y_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad \begin{vmatrix} 1 & 0 & -1 \\ 1 & 3 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \quad U_1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$\breve{H}_{01} = [1 \;\; 3 \;\; 0] = [1 \;\; 3] \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = M_o \bar{E}_o$$

$$\mu_o = 1, \quad \pi_o = 1$$

step 3, 4a, 4b

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 3 \end{vmatrix} \quad \begin{vmatrix} 0 & -2 & 0 \\ 2 & 0 & -2 \\ 0 & 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{vmatrix}$$

$$Y_3Y_2 = \begin{vmatrix} 0 & 1 \\ z & 0 \end{vmatrix} \quad \begin{vmatrix} 1 & 0 & -1 \\ 6 & -2 & -6 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 \\ 2 & 0 \end{vmatrix} \quad U_2U_1 = \begin{vmatrix} 0 & 1 \\ z & 0 \end{vmatrix}$$

3. Update

$$Y^1 = \begin{vmatrix} 0 & 1 \\ z & 0 \end{vmatrix}, \qquad U^1 = \begin{vmatrix} 0 & 1 \\ z & 0 \end{vmatrix}$$

$$y_k^1 = \begin{vmatrix} y^2_k \\ y^1_{k+1} \end{vmatrix}, \quad u_k^1 = \begin{vmatrix} u^2_k \\ u^1_{k+1} \end{vmatrix}$$

(Note: superscripts within square brackets indicate components of vectors.)

$$\tilde{F} = [0 \quad 1 \quad -3]$$

$$\tilde{G} = 1 \;, \quad U_1 u_k^o = \begin{vmatrix} u^2{}_k \\ u_k^1 \end{vmatrix} \;, \text{ so } \tilde{u}_k = \left[u_k^1\right]$$

$$\tilde{H} = \begin{vmatrix} 1 & 3 & 0 \\ 1 & 0 & -1 \end{vmatrix} \;, \quad Y_1 y_k^o = \begin{vmatrix} y_k^1 \\ y_k^2 \end{vmatrix} = \tilde{y}_k$$

2. <u>Iteration 2:</u> (i=1), step 1a, 1b

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 3 \end{vmatrix} \qquad \begin{vmatrix} 0 & -2 & 0 \\ 2 & 0 & -2 \\ 0 & 0 & 0 \end{vmatrix} \qquad \begin{vmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{vmatrix}$$

$$Y_1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad \begin{vmatrix} 6 & -2 & -6 \\ 1 & 0 & -1 \end{vmatrix} \quad \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} \qquad U_1 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

step 2b, no change

$$\tilde{H}_{11} = [1 \quad 0 \quad -1] = \left[1 \; -\tfrac{1}{3} \; -\tfrac{1}{3}\right] \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 3 \end{vmatrix} = M_1 \overline{E}_1$$

$$\mu_1 = 0 \;, \quad \pi_1 = 1$$

step 4a, 4b

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 3 \end{vmatrix} \qquad \begin{vmatrix} 0 & -2 & 0 \\ 2 & 0 & -2 \\ 0 & 0 & 0 \end{vmatrix} \qquad \begin{vmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \end{vmatrix}$$

$$Y_3 Y_2 Y_1 = \begin{vmatrix} 0 & 1 \\ z & 0 \end{vmatrix} \begin{vmatrix} 6 & -2 & -6 \\ -\frac{2}{3} & -2 & \frac{2}{3} \end{vmatrix} \begin{vmatrix} 2 & 0 \\ 2 & -\frac{1}{3} \end{vmatrix} \quad U_2 U_1 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

3. <u>Update</u>

$$Y^2 = \begin{vmatrix} z & 0 \\ 0 & z \end{vmatrix} \;, \qquad U^2 = \begin{vmatrix} 0 & 1 \\ z & 0 \end{vmatrix}$$

$$y_k^2 = \begin{vmatrix} y_{k+1}^1 \\ y_{k+1}^2 \end{vmatrix} \;, \qquad u_k^2 = \begin{vmatrix} u_k^2 \\ u_{k+1}^1 \end{vmatrix}$$

$$\tilde{F} = [0 \quad 1 \quad -3] \;, \qquad \tilde{G} = 1 \;, \qquad \tilde{u}_k = \left[u_k^1\right]$$

$$\tilde{H} = \begin{vmatrix} 1 & 3 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{vmatrix} \;, \quad Y_1 y_k^1 = \begin{vmatrix} y_{k+1}^1 \\ y^2{}_k \end{vmatrix} \;, \qquad \tilde{y}_k = \begin{vmatrix} y^1{}_k \\ y_k^2 \\ y_k^2 \end{vmatrix}$$

2. <u>Iteration 3:</u> (i=2), no change.

3. <u>Stop</u>

$$\hat{u}_k = u_k^3 = \begin{vmatrix} u_k^2 \\ u_{k+1}^1 \end{vmatrix} \;, \qquad \hat{y}_k = y_k^3 = \begin{vmatrix} y_{k+1}^1 \\ y_{k+1}^2 \end{vmatrix}$$

$$\alpha = 1 \;, \qquad \beta = 1$$

$$E_n^{-1} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} \end{vmatrix}$$

$\underline{S}^s$ $$x_{k+1} = \begin{vmatrix} 0 & -2 & 0 \\ 2 & 0 & -2 \\ \frac{2}{3} & 0 & -\frac{2}{3} \end{vmatrix} x_k + \begin{vmatrix} 2 & 0 \\ 0 & 0 \\ 0 & \frac{1}{3} \end{vmatrix} \hat{u}_k \;, \; x_o$$

$$\hat{y}_k = \begin{vmatrix} 6 & -2 & -6 \\ -\frac{2}{3} & -2 & \frac{2}{3} \end{vmatrix} x_k + \begin{vmatrix} 2 & 0 \\ 2 & -\frac{1}{3} \end{vmatrix} \hat{u}_k$$

<u>END OF ALGORITHM</u>

$\underline{S}^{-1}$ $$x_{k+1} = \begin{vmatrix} -6 & 0 & 6 \\ 2 & 0 & -2 \\ -6 & 0 & 6 \end{vmatrix} x_k + \begin{vmatrix} 1 & 0 \\ 0 & 0 \\ 1 & -1 \end{vmatrix} \hat{y}_k \;, \; x_o$$

$$\hat{u}_k = \begin{vmatrix} -3 & 1 & 3 \\ -20 & 0 & 20 \end{vmatrix} x_k + \begin{vmatrix} \frac{1}{2} & 0 \\ 3 & -3 \end{vmatrix} \hat{y}_k$$

additional conditions (3.4) $\tilde{F}x_o = -\tilde{G}\,\tilde{u}_o$:

$$[0 \quad 1 \quad -3] \; x_o = -\left[u_o^1\right]$$

subspaces of admissible initial conditions:

$$X_o = \tilde{F}^{-1} R(\tilde{G}) = R^3$$

$$X_{o/\mu} = -\tilde{F}^{+}\tilde{G}\left[u_o^1\right] + N[0 \quad 1 \quad -3]$$

$$= \begin{vmatrix} 0 \\ -\frac{1}{10} \\ \frac{3}{10} \end{vmatrix} \left[u_o^1\right] + k \begin{vmatrix} 1 & 1 \\ 0 & 3 \\ 0 & 1 \end{vmatrix}$$

additional conditions (3.7) $\tilde{y}_o = \tilde{H}x_o$

$$\begin{vmatrix} y_o^1 \\ y_o^2 \\ y_o^2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{vmatrix} x_o$$

6. CONCLUSIONS

The Luenberger shuffle algorithm and the Silverman structure algorithm were combined into a descriptor system structure algorithm (DSSA). DSSA was used to extend known results on state-space system left and right inversion to descriptor systems.

On further study, DSSA should yield results on the structure and properties of descriptor systems comparable to those of Silverman, Molinari, and others for the state-space case [4,8,15,28-30]. Results on descriptor system unknown input unobservable subspaces, feedback properties, optimal control, etc. should be forthcoming.

REFERENCES

[1] D. G. Luenberger, "Dynamic equations in descriptor form," IEEE Trans. Automat. Contr., pp. 312-321, June 1977.

[2] A. Fettweis, "On the algebraic derivation of state equations," IEEE Trans. Circuit Theory, vol. CT-16, no. 2, pp. 171-175, May 1969.

[3] F. R. Gantmacher, The Theory of Matrices, New York: Chelsea, 1974.

[4] P. M. Van Dooren, "The generalized eigenstructure problem in linear system theory," IEEE Trans. Automat. Contr., vol. AC-26, no. 1, pp. 111-129, Feb. 1981.

[5] V. C. Klema and A. J. Laub, "The singular value decomposition: its computation and some applications," IEEE Trans. Automat. Contr., vol. AC-25, no. 2, pp. 164-176, April 1980.

[6] D. G. Luenberger, "Time-invariant descriptor systems," Automatica, vol. 14, pp. 473-480, 1978.

[7] L. M. Silverman, "Inversion of multivariable linear systems," IEEE Trans. Automat. Contr., vol. AC-15, no. 3, pp. 270-276, June 1969.

[8] L. M. Silverman, "Discrete Riccati equations: alternative algorithms, asymptotic properties, and system theory interpretations," Control and Dynamic Systems, vol. 12, C. T. Leondes ed., New York: Academic Press, pp. 313-386, 1976.

[9] H. H. Rosenbrock, State-Space and Multivariable Theory, New York: Wiley, 1970.

[10] V. Lovass-Nagy, R. J. Miller, and D. L. Powers, "An introduction to the application of the simplest matrix-generalized inverse in systems science," IEEE Trans. Circuits and Systems, vol. CAS-25, no. 9, pp. 766-771, Sept. 1978.

[11] C. R. Rao and S. K. Mitra, Generalized Inverse of Matrices and Its Applications, New York: Wiley, 1971.

[12] A. Ben-Israel and T. N. E. Greville, Generalized Inverses: Theory and Applications, New York: Krieger, 1980.

[13] J. L. Massey and M. K. Sain, "Inverses of linear sequential circuits," IEEE Trans. Computers, pp. 330-337, April 1968.

[14] G. Basile and G. Marro, "On the perfect output controllability of linear dynamic systems," Richerche di Automatica, vol. 2, no. 1, pp. 1-10, Jan. 1971.

[15] L. M. Silverman and H. J. Payne, "Input-output structure of linear systems with application to the decoupling problem," SIAM J. Contr., vol. 9, pp. 199-233, May 1971.

[16] P. Dorato, "On the inverse of linear dynamical systems," IEEE Trans. Sys. Science and Cyber., vol. SSC-5, no. 1, pp. 43-48, Jan. 1969.

[17] F. Lewis, "Generalized output-nulling subspaces: Riccati equation computation and applications," IEEE Trans. Automat. Contr., to appear Dec. 1982.

[18] L. M. Silverman, "Properties and applications of inverse systems," IEEE Trans. Automat. Contr., pp. 436-438, Aug. 1968.

[19] R. W. Brockett, "Poles zeros, and feedback: state space interpretation," IEEE Trans. Automat. Contr., vol. AC-10, pp. 129-135, April 1965.

[20] G. Basile and G. Marro, "Controlled and conditioned invariant subspaces in linear system theory," JOTA, vol. 3, no. 5, pp. 306-315, 1969.

[21] M. K. Sain and J. L. Massey, "Invertibility of time-invariant dynamical systems," IEEE Trans. Automat. Contr., pp. 141-149, April 1969.

[22] S. P. Singh, "A note on inversion of linear systems," IEEE Trans. Automat. Contr., pp. 492-493, Aug. 1970.

[23] P. J. Antsaklis, "Stable proper n-th order inverses," IEEE Trans. Automat. Contr., vol. AC-23, no. 6, pp. 1104-1106, December 1978.

[24] L. M. Silverman, "Comments on 'Stable proper n-th order inverses'," IEEE Trans. Automat. Contr., vol. AC-24, no. 4, pp. 673-674, August, 1979.

[25] P. J. Moylan, "Stable inversion of linear systems," IEEE Trans. Automat. Contr., pp. 74-78, February 1977.

[26] B. Dziurla and R. Newcomb, "The Drazin inverse and semi-state equations," Proc. 4th Int. Symp. Mathematical Theory of Networks and Systems, Delft, The Netherlands, pp. 283-289, July 1979.

[27] E. L. Yip and R. F. Sincovec, "Solvability, controllability, and observability of continuous descriptor systems," IEEE Trans. Automatic Contr., pp. 702-707, June 1981.

[28] B. P. Molinari, "A strong controllability and observability in linear multivariable control," IEEE Trans. Automat. Contr., pp. 761-764, October 1976.

[29] B. P. Molinari,"Extended controllability and observability for linear systems," IEEE Trans. Automat. Contr., pp. 136-137, February 1976.

[30] B. P. Molinari, "Structural invariants of linear multivariable systems," Int. J. Control, vol. 28, no. 4, pp. 493-510, 1978.

[31] K-T Wong, "The eigenvalue problem $\lambda Tx + Sx$," J. Diff. Equations, vol. 16, pp. 270-281, 1974.

[32] G. C. Verghese, "Further Notes on Singular Descriptions," Proc. JACC, paper TA-4, June 1981.

[33] S. L. Campbell, C. D. Meyer Jr., and N. J. Rose, "Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients," SIAM J. Appl. Math, vol. 31, no. 3, pp. 411-425, November 1976.

Proceedings of the
1983 AMERICAN CONTROL CONFERENCE

Sheraton-Palace Hotel, San Francisco, California

June 22, 23, 24, 1983

The American Automatic Control Council

AIAA
AIChE
ASME
IEEE
ISA

Wednesday, June 22, 1983
Sheraton-Palace Hotel
San Francisco, California

ACC

83CH1915-8

Descriptor Systems: Decomposition Into Forward and Backward Subsystems

FRANK L. LEWIS

Abstract—It is shown that a decomposition of the descriptor space with certain properties induces a decomposition of the descriptor system into forward and backward subsystems. Different decompositions correspond to different apportionments of the required boundary conditions between initial and final times. Applications include several different but equivalent methods for solving the descriptor system and a clarification of the relations between some previously known methods of solution.

I. Introduction

There are many algorithms to solve descriptor or generalized-state space systems [2]–[9]. We present an approach which attempts to interrelate some of these algorithms by dealing with the geometric structure of the descriptor space. Given a decomposition of the descriptor space with certain properties $\mathscr{S} \oplus \mathscr{T} = \mathscr{R}^n$, we show that a decomposition of the descriptor system into forward and backward subsystems is induced. The subspaces $\mathscr{S}$ and $\mathscr{T}$ can be chosen depending on how it is desired to split up between initial and final times the additional conditions required to specify a unique solution of the descriptor equation [2].

Manuscript received October 12, 1982; revised February 23, 1983. This work was supported by the National Science Foundation under Contract ECS-8204656.

The author is with the School of Electrical Engineering, Georgia Institute of Technology, Atlanta, GA 30332.

0018-9286/84/0200-0167$01.00 ©1984 IEEE

We present several equivalent formulations of our forward/backward (F/B) decomposition. These reduce in particular cases to several known algorithms. The F/B decomposition thus helps clarify the relation between these algorithms. It also provides several equivalent methods for solving descriptor systems and yields expressions for the subspaces of admissible initial [5], [6], [8], [14] and final conditions. Light is shed by this approach on the nomenclature "nonoriented abstract object" used in [20], [21].

The key paper by Wong [4] provides the background for this note.

II. The Forward/Backward Decomposition

Consider the following linear discrete time invariant dynamical equation over the real numbers $\mathscr{R}$:

$$Ex_{k+1} = Ax_k + Bu_k; \qquad k = 0,1,\cdots,N-1 \tag{2.1}$$

where the $u_k \in \mathscr{R}^m$, $x_k \in \mathscr{R}^n$, and E, A, B are constant matrices of appropriate dimension. We consider the case $(zE - A)$ regular, i.e., $|zE - A|$ not identically zero [1]–[3]. Luenberger has shown that in this case a solution sequence x_k exists for all sequences u_k [2]. N simply specifies a time interval of interest.

In the event of singular E, x_k should not be considered a state for (2.1) and following Luenberger [2] we call x_k the *descriptor variable*. If E is nonsingular, then (2.1) is a state system which can be solved by iterating forward in time given initial condition x_0. On the other hand, if A is nonsingular, then (2.1) can be solved by iterating backward in time given final condition x_N.

If E and A are both singular, the situation is considerably more complex and interesting. There are many different approaches for solving (2.1) in this case [2]–[9]. Luenberger has shown that if $(zE - A)$ is regular, then a unique solution sequence x_k is determined by specifying u_k and additional conditions x_k at the initial and final times $k = 0$ and $k = N$.

In general, the additional conditions x_o and x_N cannot be specified in an arbitrary manner, and the subject of "admissible additional conditions" has received much attention from many different points of view [2]–[9].

We propose to decompose (2.1) into two subsystems, which we call the *forward* and *backward subsystems*. This approach makes clear exactly which additional conditions x_o, x_N are admissible and yields different decompositions depending on how it is desired to specify x_o, x_N. We show how several of the approaches [2]–[9] follow as special cases of our descriptor system forward/backward (F/B) decomposition. Our point of view thus relates several approaches which seem at first glance to have little to do with each other.

Subspace $\mathscr{S} \subset \mathscr{R}^n$ is a *deflating subspace* [10] if $\dim(E\mathscr{S} + A\mathscr{S}) = \dim(\mathscr{S})$. If $E = I$, the identity, then this reduces to the definition for an A-invariant subspace [11]. Let $R(A), N(A)$ denote the range and nullspace, respectively, of A. Then we have the following result. The proof follows directly from the definitions of regularity and deflating subspace and is omitted.

Lemma 2.1: Let $(zE - A)$ be regular, $\mathscr{S}$ and $\mathscr{T} \subset \mathscr{R}^n$ be deflating subspaces with $\mathscr{S} \oplus \mathscr{T} \mathscr{R}^n$. Then $(E\mathscr{S} + A\mathscr{S}) \oplus (E\mathscr{T} + A\mathscr{T}) = \mathscr{R}^n$. □

Now let $\mathscr{S}$ and $\mathscr{T}$ be deflating subspaces with $\mathscr{S} \oplus \mathscr{T} = \mathscr{R}^n$. Define $\bar{\mathscr{S}} = E\mathscr{S} + A\mathscr{S}$, $\bar{\mathscr{T}} = E\mathscr{T} + A\mathscr{T}$. Then by the above result $\bar{\mathscr{S}} \oplus \bar{\mathscr{T}} = \mathscr{R}^n$. Transform (2.1) to bases adapted to the decompositions $\mathscr{S} \oplus \mathscr{T} = \mathscr{R}^n$, $\bar{\mathscr{S}} \oplus \bar{\mathscr{T}} = \mathscr{R}^n$: that is, define matrices $\bar{V}, \tilde{V}, \bar{U}, \tilde{U}$ so that $\mathscr{S} = R(\bar{V})$, $\mathscr{T} = R(\tilde{V})$, $\bar{\mathscr{S}} = R(\bar{U})$, $\bar{\mathscr{T}} = R(\tilde{U})$ and let $V = [\bar{V} \;\; \tilde{V}]$, $U = [\bar{U} \;\; \tilde{U}]$. Then

$$U^{-1}[zE - A|B]V = \left[\begin{array}{cc|c} z\bar{E} - \bar{A} & 0 & \bar{B} \\ 0 & z\tilde{E} - \tilde{A} & \tilde{B} \end{array}\right]. \tag{2.2}$$

[Matrices $\bar{E}, \bar{A}, \bar{B}, \tilde{E}, \tilde{A}, \tilde{B}$ are defined by (2.2).] This transformation corresponds to a descriptor-space transformation $T = V^{-1}$. We will use the caret to denote quantities represented in the new basis, hence

$$\hat{x}_k = Tx_k = V^{-1}x_k \tag{2.3}$$

represents the descriptor-space transformation. Our procedure should be compared to discussions in [10].

At this point it is not clear what we have gained. We have decomposed (2.1) into two descriptor subsystems (2.2), but in general matrices $\bar{E}, \bar{A}, \tilde{E}, \tilde{A}$ are all singular; so that the problems associated with solving (2.1) also arise in solving the two subsystems. Let us therefore pose restrictions on $\mathscr{S}$ and $\mathscr{T}$ so that $\bar{E}, \tilde{A}$ become nonsingular.

To this end, we use two algorithms which have appeared before [4]. Let $\mathscr{X}_k, \mathscr{Y}_k \subset \mathscr{R}^n$, and let

$$\mathscr{X}_{k+1} = A^{-1}(E\mathscr{X}_k), \qquad \mathscr{X}_0 = \mathscr{R}^n, \tag{2.4a}$$

$$\mathscr{Y}_k = E^{-1}(A\mathscr{Y}_{k+1}), \qquad \mathscr{Y}_n = \mathscr{R}^n \tag{2.4b}$$

for $k = 0,1,\cdots,n-1$. The notation $A^{-1}(\cdot)$ indicates inverse image.

Define the *initial manifold* $H_I = \mathscr{X}_n$ [4] and the *final manifold* $H_F = \mathscr{Y}_o$. Note that H_I corresponds to Verghese's (E and A,0)-invariant subspace [14]. Note further that $AH_I \subset EH_I$ and $EH_F \subset AH_F$; so that H_I and H_F are both deflating subspaces (it is shown in [4] that $N(E) \cap H_I = 0$, and similarly $N(A) \cap H_F = 0$). Now we have the next result.

Theorem 2.2: Let $(zE - A)$ be regular, $\mathscr{S}$ and $\mathscr{T} \subset \mathscr{R}^n$ be deflating subspaces with $\mathscr{S} \oplus \mathscr{T} = \mathscr{R}^n$. In addition, suppose $\mathscr{S} \subset H_I$, $\mathscr{T} \subset H_F$. Then in bases adapted to the decompositions $\mathscr{S} \oplus \mathscr{T} = \mathscr{R}^n$, $\bar{\mathscr{S}} \oplus \bar{\mathscr{T}} = \mathscr{R}^n$ system (2.1) becomes (2.2) with $\bar{E}$ and $\tilde{A}$ nonsingular.

Proof: (In the full notation of [11]) $\bar{E}$ is the restriction of E to $\mathscr{S}$ with codomain $\bar{\mathscr{S}}$, $\bar{\mathscr{S}}|E|\mathscr{S}$. It is shown in [4] that $E|H_I$ is one-to-one if and only if $(zE - A)$ is regular. Hence, $E|\mathscr{S}$, $\mathscr{S} \subset H_I$ is one-to-one. Therefore, $\bar{\mathscr{S}}|E|\mathscr{S}$ is also. Similarly, $\tilde{A}$ is $\bar{\mathscr{T}}|A|T$, which is one-to-one if $(zE - A)$ is regular. Since $\bar{E}$ and $\tilde{A}$ are square due to their restricted codomains, they are both nonsingular. □

Theorem 2.2 allows us to write the following *forward/backward* (F/B) *decomposition* for (2.1) which is induced by the selected decomposition $\mathscr{S} \oplus \mathscr{T} = \mathscr{R}^n$ of the descriptor space. Let

$$\hat{x}_k = Tx_k = \begin{bmatrix} \hat{x}_k^f \\ \hat{x}_k^b \end{bmatrix}$$

where $\hat{x}_k^f \in \mathscr{S} \subset H_I$, $\hat{x}_k^b \in \mathscr{T} \subset H_F$. Then (2.2) yields

$$\hat{x}_{k+1}^f = \bar{E}^{-1}\bar{A}\hat{x}_k^f + \bar{E}^{-1}\bar{B}u_k, \qquad \hat{x}_0^f \in S \tag{2.5a}$$

$$\hat{x}_k^b = \tilde{A}^{-1}\tilde{E}\hat{x}_{k+1}^b - \tilde{A}^{-1}\tilde{B}u_k, \qquad \hat{x}_N^b \in \mathscr{T} \tag{2.5b}$$

for $k = 0,1,\cdots,N-1$.

System (2.5a) is a *forward subsystem* which can be solved by iterating forward in time given initial conditions $\hat{x}_o^f$, and (2.5b) is a *backward subsystem* which can be solved by iterating backward in time given final conditions $\hat{x}_N^b$.

Let $\langle A|B\rangle_N = R(B) + AR(B) + \cdots + A^{N-1}R(B)$. Then from (2.5) we can write the *subspace of admissible initial conditions* given the decomposition $\mathscr{S} \oplus \mathscr{T} = \mathscr{R}^n$ as

$$\hat{\mathscr{X}}_0 = \mathscr{S} \oplus \left((\tilde{A}^{-1}\tilde{E})^N \mathscr{T} + \langle \tilde{A}^{-1}\tilde{E} | \tilde{A}^{-1}\tilde{B} \rangle_N \right). \tag{2.6a}$$

Compare this in the case $\mathscr{S} = H_I$, $\mathscr{T} = H_{NI}$ (defined in Section III) with the results in [5], [6], [8], [14]. Similarly, we can write the *subspace of admissible final conditions* given the decomposition $\mathscr{S} \oplus \mathscr{T} = \mathscr{R}^n$ as

$$\hat{\mathscr{X}}_N = \mathscr{T} \oplus \left((\bar{E}^{-1}\bar{A})^N \mathscr{S} + \langle \bar{E}^{-1}\bar{A} | \bar{E}^{-1}\bar{B} \rangle_N \right). \tag{2.6b}$$

Our approach shows that we may exercise our option for selecting the additional conditions required to specify a unique solution in several ways. If we desire, given $\mathscr{S}$ and $\mathscr{T}$, to specify as many initial conditions as possible, then we restrict the choice for sequence u_k, which according to (2.6a) must be chosen to make the desired $\hat{x}_o \in \hat{\mathscr{X}}_o$ compatible with (2.1). Similarly, if we desire, given $\mathscr{S}$ and $\mathscr{T}$, to specify as many final conditions as possible, then according to (2.6b) we must again restrict the choice of u_k to values that make $\hat{x}_N \in \hat{\mathscr{X}}_N$ compatible. An alternative selection of the additional conditions which does not restrict the choice for u_k is the choice of the additional conditions to lie in the *subspace of admissible split conditions*

$$\hat{\mathscr{X}}_{ON} = \left\{ \begin{bmatrix} \hat{x}_o^f \\ \hat{x}_N^b \end{bmatrix} : \hat{x}_o^f \in \mathscr{S}, \hat{x}_N \in \mathscr{T} \right\}. \tag{2.6c}$$

In addition to this freedom in choosing $\hat{x}_o, \hat{x}_N$ we have of course the freedom of selecting the subspace $\mathscr{S}$ and $\mathscr{T}$ to begin with, which we illustrate in the next section.

III. Maximally Forward and Maximally Backward Decompositions

In this section we consider two particular choices for the pair $\mathscr{S}, \mathscr{T}$ and relate our result (2.2)–(2.5) to known results.

First, we consider the following choice. Define $H_{NI} = \mathscr{Y}_o$ given by (2.4b) when $\mathscr{Y}_n = 0$.

Theorem 3.1: Let $\mathscr{S} = H_I$ and $\mathscr{T} = H_{NI}$. Then $\mathscr{S} \oplus \mathscr{T} = \mathscr{R}^n$, $\mathscr{T} \subset H_F$ so that (2.5) is a decomposition for (2.1). Furthermore,

a) $\tilde{A}^{-1}\tilde{E}$ is nilpotent.

b) $\mathscr{S}$ is the maximal subspace such that $\bar{E} = \bar{\mathscr{S}}|E|\mathscr{S}$ is nonsingular.

Proof: Note that $\mathscr{T} = H_N$ as defined in [4] and use the results there. □

Selecting $\mathscr{S} = H_I$, $\mathscr{T} = H_{NI}$ as in Theorem 3.1 results in a *maximally forward decomposition* of (2.1) and makes (2.2) the Kronecker canonical decomposition [1], [12], [13], [17] for (2.1). In this case (2.5) is the system of [6], for which a closed-form solution is given therein. See also [22].

Now, consider the following choice for $\mathscr{S}, \mathscr{T}$. Define $H_{NF} = \mathscr{X}_n$ given by (2.4a) when $\mathscr{X}_o = 0$.

Theorem 3.2: Let $\mathscr{T} = H_F$ and $\mathscr{S} = H_{NF}$. Then $\mathscr{S} \oplus \mathscr{T} = \mathscr{R}^n$ and $\mathscr{S} \subset H_I$ so that (2.5) is a decomposition for (2.1). Furthermore,

a) $\bar{E}^{-1}\bar{A}$ is nilpotent,

b) $\mathscr{T}$ is the maximal subspace such that $\tilde{A} = \tilde{\mathscr{T}}|A|\mathscr{T}$ is nonsingular.

Proof: Similar to Theorem 3.1. □

Selecting $\mathscr{S} = H_{NF}$, $\mathscr{T} = H_F$ as above results in a *maximally backward decomposition* of (2.1). A closed-form solution similar to the one given in [6] can be written in this case. In fact, the F/B decomposition (2.5) allows us to write closed-form expressions for the solution $\hat{x}_k$ in terms of u_k, $\hat{x}_o^f$, and $\hat{x}_N^b$ for any choice of $\mathscr{S}$ and $\mathscr{T}$. The results of Sections IV and V lead to closed-form expressions for the descriptor variable x_k in the original coordinates, although we shall not pursue this subject in this note.

Note that the maximally forward decomposition has the maximal number ($\deg(|zE - A|)$) of additional conditions specified at time $k = 0$, while the maximally backward decomposition has the maximal number ($\deg(|zE - A|)$) of additional conditions specified at time $k = N$. See the discussion in [15]. See also [19] which expresses H_I, H_{NI}, H_F, H_{NF} in terms of eigenspaces of $(\lambda E - A)^{-1}E$.

In Section VI we present an example illustrating these concepts.

IV. Double-Sweep Formulation of F/B Decomposition

In [2], Luenberger gives a "double-sweep" method of solution for (2.1). Although the exact relationship of his method to our approach is still under investigation, we can present a double-sweep method based on (2.5) which is similar to his.

Thus, we identify $\hat{x}_k^f$ in (2.5a) with his "condition vector"; so that (2.5a) becomes the initial forward sweep of the algorithm. That is, given $\hat{x}_o^f$ we can find $\hat{x}_k^f$ for all k. Now we must derive a backward sweep which reconstructs x_k given $\hat{x}_k^f$, u_k, and $\hat{x}_N^b$ by iterating backward in time.

To this end, note that $x_k = T^{-1}\hat{x}_k = \bar{V}\hat{x}_k^f + \tilde{V}\hat{x}_k^b$, or using (2.5b) and the fact that $\hat{x}_{k+1}^b = \tilde{T}x_{k+1}$,

$$x_k = \bar{V}\hat{x}_k^f + \tilde{V}\tilde{A}^{-1}\tilde{E}\tilde{T}x_{k+1} - \tilde{V}\tilde{A}^{-1}\tilde{B}u_k, \qquad k = 0, 1, \cdots, N-1.$$

$$x_N = \bar{V}\hat{x}_N^f + \tilde{V}\hat{x}_N^b. \tag{4.1}$$

Equations (2.5a) and (4.1) are very similar to those presented in [2] as the double-sweep method.

The double-sweep F/B method of solution of (2.1) is illustrated in the example in the next section.

V. Example

This example is taken from [5]. Let (2.1) be given by

$$\begin{bmatrix} 1 & 0 & -2 \\ -1 & 0 & 2 \\ 2 & 3 & 2 \end{bmatrix} x_{k+1} = \begin{bmatrix} 0 & -1 & -2 \\ 27 & 22 & 17 \\ -18 & -14 & -10 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} u_k.$$

Then, using (2.4a) and (2.4b), we obtain

$$H_I = R\begin{bmatrix} 1 & 1 \\ 8 & -2 \\ -13 & 1 \end{bmatrix}, \qquad H_F = R\begin{bmatrix} 1 & 2 \\ 8 & -2 \\ -13 & 1 \end{bmatrix},$$

$$H_{NI} = R\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \qquad H_{NF} = R\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Note $H_{NI} \subset H_F$, $H_{NF} \subset H_I$. (In fact, these subspaces were found not from (2.4a), (2.4b) but by using the generalized eigenstructure of $(\lambda E - A)$ [4], [10]. See [16] for details.)

1) Maximally Forward Decomposition: Let $\mathscr{S} = H_I$, $\mathscr{T} = H_{NI}$. Thus,

$$V = T^{-1} = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 8 & -2 & -2 \\ -13 & 1 & 1 \end{array}\right],$$

$$\bar{\mathscr{S}} = E\mathscr{S} + A\mathscr{S} = R\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 2 \end{bmatrix},$$

$$\tilde{\mathscr{T}} = E\mathscr{T} + A\mathscr{T} = R\begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix},$$

and

$$\mathscr{U} = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 3 \\ 0 & 2 & -2 \end{array}\right].$$

Therefore, (2.5) becomes

$$\hat{x}_{k+1}^f = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 0 \end{bmatrix}\hat{x}_k^f + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u_k, \qquad \hat{x}_o^f \text{ given}$$

$$\hat{x}_k^b = -\frac{1}{9}u_k, \qquad \hat{x}_N^b \text{ given.}$$

2) Maximally Backward Decomposition: Let $\mathscr{S} = H_{NF}$, $\mathscr{T} = H_F$. Then

$$V = T^{-1} = \left[\begin{array}{c|cc} 1 & 1 & 2 \\ -2 & 8 & -2 \\ 1 & -13 & 1 \end{array}\right],$$

$$\bar{\mathscr{S}} = E\mathscr{S} + A\mathscr{S} = R\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix},$$

$$\tilde{\mathscr{T}} = E\mathscr{T} + A\mathscr{T} = R\begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 0 & -2 \end{bmatrix},$$

and

$$U = \left[\begin{array}{c|cc} -1 & 1 & 0 \\ 1 & -1 & 3 \\ -2 & 0 & -2 \end{array}\right].$$

Therefore, (2.5) becomes

$$\hat{x}_{k+1}^f = -u_k, \qquad \hat{x}_o^f \text{ given}$$

$$\hat{x}_k^b = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 0 \end{bmatrix}\hat{x}_{k+1}^b - \begin{bmatrix} 0 \\ \frac{1}{9} \end{bmatrix} u_k, \qquad \hat{x}_N^b \text{ given.}$$

3) Double-Sweep Formulation of Maximally Forward Decomposition: Using $\mathscr{S} = H_I$, $\mathscr{T} = H_{NI}$, then (2.5a) and (4.1) yield

$$\hat{x}_{k+1}^f = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & 0 \end{bmatrix}\hat{x}_k^f + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u_k, \qquad \hat{x}_o^f \text{ given}$$

$$x_k = \begin{bmatrix} 1 & 1 \\ 8 & -2 \\ -13 & 1 \end{bmatrix}\hat{x}_k^f - \frac{1}{9}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} u_k, \qquad k = 0, 1, \cdots, N-1,$$

$$x_N = \bar{V}\hat{x}_N^f + \tilde{V}\hat{x}_N^b = \begin{bmatrix} 1 & 1 \\ 8 & -2 \\ -13 & 1 \end{bmatrix} \hat{x}_N^f + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \hat{x}_N^b.$$

4) Discussion: Note that in 1) the F/B decomposition has a *forward state* of maximal dimension equal to $\deg(|zE - A|) = 2$. In 2) the F/B decomposition has a *backward state* of this maximal dimension.

1) and 3) yield different but equivalent formulations of the same maximally forward F/B decomposition, and either of them can be used to solve (2.1) for x_k.

VII. Conclusions

Given a decomposition of the descriptor space $\mathcal{S} \oplus \mathcal{T} = \mathcal{R}^n$ where $\mathcal{S}$, $\mathcal{T}$ are deflating subspaces contained, respectively, in the initial manifold and final manifold, we derived a decomposition of the descriptor system into forward and backward subsystems. We presented several equivalent formulations for this F/B decomposition, and interrelated several known methods for solving descriptor systems. We presented expressions for subspaces of admissible initial, final, and split conditions. Our approach yields some new relationships between several known results and provides several methods of solution for descriptor systems.

Since the subspaces discussed in this note are functions purely of the matrices E and A, the results will extend to the continuous case. For example, if the initial condition is contained in the subspace H_I, then impulsive behavior does not occur in the absence of impulsive inputs [28].

References

[1] F. R. Gantmacher, *The Theory of Matrices*. New York: Chelsea, 1974.

[2] D. G. Luenberger, "Dynamic equations in descriptor form," *IEEE Trans. Automat. Contr.*, vol. AC-22, pp. 312–321, June 1977.

[3] D. G. Luenberger, "Time-invariant descriptor systems," *Automatica*, vol. 14, pp. 473–480, 1978.

[4] K-T Wong, "The eigenvalue problem $\lambda Tx + Sx$," *J. Diff. Equations*, vol. 16, pp. 270–281, 1974.

[5] S. L. Campbell, C. D. Meyer Jr., and N. J. Rose, "Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients," *SIAM J. Appl. Math.*, vol. 31, pp. 411–425, Nov. 1976.

[6] E. L. Yip and R. F. Sincovec, "Solvability, controllability, and observability of continuous descriptor systems," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 702–707, June 1981.

[7] A. Fettweis, "On the algebraic derivation of state equations," *IEEE Trans. Circuit Theory*, vol. CT-16, no. 2, pp. 171–175, May 1969.

[8] F. L. Lewis, "Inversion of descriptor systems," in *Proc. ACC*, San Francisco, CA, June 1983.

[9] R. F. Sincovec, A. M. Erisman, E. L. Yip, and M. A. Epton, "Analysis of descriptor systems using numerical algorithms," *IEEE Trans. Automat. Contr.*, vol. AC-26, no. 1, pp. 139–147, Feb. 1981.

[10] P. M. Van Dooren, "The generalized eigenstructure problem in linear system theory," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 111–129, Feb. 1981.

[11] W. M. Wonham, *Linear Multivariable Control: A Geometrical Approach*, 2nd ed. New York: Springer-Verlag, 1979.

[12] H. H. Rosenbrock, *State-Space and Multivariable Theory*. New York: Wiley, 1970.

[13] J. H. Wilkinson, "Linear differential equations and Kronecker's canonical form," in *Recent Advances in Numerical Analysis*, C. de Boor and G. H. Golub, Eds. New York: Academic, 1978.

[14] G. C. Verghese, "Further notes on singular descriptions," in *Proc. JACC*, June 1981, paper TA-4.

[15] F. L. Lewis, "Descriptor systems: Expanded descriptor equation and Markov parameters," *IEEE Trans. Automat. Contr.*, vol. AC-28, June 1983.

[16] F. L. Lewis and K. Ozcaldiran, "The generalized eigenstructure problem and descriptor systems," presented at the SIAM National Meeting, Denver, CO, 1983.

[17] H. H. Rosenbrock, "Structural properties of linear dynamical systems," *Int. J. Contr.*, vol. 20, no. 2, pp. 191–202, 1974.

[18] J. D. Cobb, "Descriptor variable and generalized singularly perturbed systems: A geometric approach," Ph.D. dissertation, Dept. Elec. Eng., Univ. Illinois, Urbana-Champaign, 1980.

[19] D. Cobb, "Feedback and pole placement in descriptor variable systems," *Int. J. Contr.*, vol. 33, no. 6, pp. 1135–1146, 1981.

[20] J. D. Aplevich, "Time-domain input-output representations of linear systems," *Automatica*, vol. 17, no. 3, pp. 509–522, 1981.

[21] L. A. Zadeh and C. A. Desoer, *Linear System Theory*. New York: McGraw-Hill, 1963.

[22] C. E. Langenhop, "Controllability and stabilizability of regular singular linear systems with constant coefficients," Dept. of Mathematics, S. Illinois Univ., Dec. 6, 1979.

[23] S. L. Campbell, *Singular Systems of Differential Equations*. New York: Pitman, 1980.

[24] S. L. Campbell, *Singular Systems of Differential Equations II*. New York, Pitman, 1982.

[25] L. Pandolfi, "Controllability and stabilization for linear systems of algebraic and differential equations," *JOTA*, vol. 30, pp. 601–620, 1980.

[26] L. Pandolfi, "On the regular problem for linear degenerate control systems," *JOTA*, vol. 33, pp. 241–254, 1981.

[27] J. H. Wilkinson, "Note on the practical significance of the Drazin inverse," in *Recent Applications of Generalized Inverses*. New York: Pitman, 1982.

[28] G. C. Verghese, B. C. Levy and T. Kailath, "A generalized state-space for singular systems," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 811–831, Aug. 1981.

0018-9286/84/0200-0170$01.00 ©1984 IEEE

REACHABILITY AND CONTROLLABILITY FOR DESCRIPTOR SYSTEMS

F. L. Lewis and K. Ozcaldiran

School of Electrical Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332

ABSTRACT

Two properties, reachability and modal controllability, are discussed for descriptor systems. The first is associated with open-loop control and the second with closed-loop control.

The open-loop control problem is solved for descriptor systems, and a result by Moore on closed-loop eigenstructure assignability is generalized to the descriptor case.

I. INTRODUCTION

Consider the time-invariant descriptor system

$$E\dot{x}(t) = Ax(t) + Gu(t) \tag{1.1}$$

where $x(t) \in R^n$, $u(t) \in R^m$ and E, A, and G are real matrices with appropriate dimensions. We shall assume that (i) E is singular, (ii) $|\lambda E-A|$, as a polynomial in λ, is not identically zero (i.e. $(\lambda E-A)$ is regular), and (iii) $u(t)$ is differentiable sufficiently many times.

Recently, a considerable amount of work on descriptor systems has appeared in the literature. The interested reader is referred to [23] for an extensive survey of the existing results.

In this paper we discuss the notions of reachability and controllability for (1.1) and relate various definitions of previous authors. We present a solution to the open-loop control problem, which depends on the notion of reachability. Then we generalize a result on closed-loop eigenstructure assignability by Moore [20] to the case $|E| = 0$. It is known [21] that pole assignability depends on the notion of controllability.

It is clear from our results that if $|E| = 0$ then the existence of an open-loop control is not equivalent to closed-loop eigenstructure assignability. This is in contrast to the state-space situation.

We begin with a review of the relative eigenstructure of the pair (E,A).

2. RELATIVE EIGENSTRUCTURE

Let E and A be nxn real matrices and define their relative spectrum $\sigma(E,A)$ as those complex numbers λ for which

$$(\lambda E-A)v = 0 \tag{2.1}$$

for some nonzero $v \in C^n$. Zero is in $\sigma(E,A)$ if and only if A is singular, for then $Av = 0$ for some nonzero v. Infinity is in $\sigma(E,A)$ if and only if E is singular for then, taking the limit in (2.1), $Ev = 0$ for some nonzero.

We define the finite relative eigenstructure as follows.

For each distinct finite element λ_i of $\sigma(E,A)$, define (finite) rank 1 relative eigenvectors v_i^1 by

$$(\lambda_i E-A)v_i^1 = 0 . \tag{2.2a}$$

For each independent solution v_{ij}^1 of this equation, define rank k relative eigenvectors v_{ij}^k as the linearly independent vectors generated by the recursion

$$(\lambda_i E-A)v_{ij}^{k+1} = -Ev_{ij}^k , \quad k > 1 . \tag{2.2b}$$

Thus each rank 1 eigenvector for finite $\lambda_i \in \sigma(E,A)$ has an associated relative eigenvector chain $\{v_{ij}^1, v_{ij}^2, \ldots\}$.

In the event E is nonsingular the above finite relative eigenstructure reverts to the standard eigenstructure of the single matrix $E^{-1}A$. If E is singular we must define in addition the infinite relative eigenstructure as follows.

Define (infinite) rank 1 relative eigenvectors v_∞^1 by

$$Ev_\infty^1 = 0 . \tag{2.3a}$$

(These are just the rank 1 eigenvectors for eigenvalue zero of the single matrix E.) There are $\eta \triangleq \dim N(E)$ independent solutions to (2.3a), where $N(\cdot)$ represents nullspace. For each independent solution $v_{\infty j}^1$ define rank k relative eigenvectors at ∞, $v_{\infty j}^k$, as the linearly independent vectors generated by

$$Ev_{\infty j}^{k+1} = Av_{\infty j}^k , \quad k > 1 . \tag{2.3b}$$

Thus each rank 1 eigenvector at infinity has an associated relative eigenvector chain $\{v_{\infty j}^1, v_{\infty j}^2, \ldots\}$.

There are η chains at ∞. Define α_i as the length of (i.e. number of eigenvectors in) the i^{th} chain at ∞. Then $\nu \triangleq \sup\{\alpha_i\}$ is the index (of nilpotency) of the matrix pencil $(sE-A)$ [1]. Assume without loss of generality at $\nu = \alpha_1 > \alpha_2 > \dots > \alpha_t > \alpha_{t+1} = \dots = \alpha_n = 1$

References [2-6] either define or make use of the relative eigenstructure of matrices.

Let H_I = span $\{v^k_{ij}\}$, H_N = span $\{v^k_{\infty j}\}$. (If v^k_{ij} ($v^k_{\infty j}$) is complex, we take its real and imaginary parts as basis vectors in order that H_I (H_N) be a real space.) Then

$$R^n = H_I \oplus H_N . \qquad (2.4)$$

We call these subspaces the <u>finite</u> and <u>infinite relative eigenspace</u> respectively. H_I has also been called the subspace of admissible initial conditions for the system (1.1). See [7-11]. See also [6,22] where H_I and H_N are interpreted respectively as slow and fast subspaces for (1.1). Define

$$\rho \triangleq \sum_{i=1}^{n} \alpha_i ,$$

the total number of eigenvectors at ∞. Then dim $H_N = \rho$, dim $H_I = n-\rho$. Note that $N(E) \subset H_N$. Note also that degree of $|\lambda E-A|$ is $n-\rho$.

By selecting the relative eigenvectors as a basis, the pencil $(\lambda E-A)$ can be transformed to a normal form. To wit, let <u>right modal matrix</u> V have as columns the finite eigenvectors v^k_{ij} followed by the infinite eigenvectors $v^k_{\infty j}$. Order these vectors so that the index set $\{i,j,k\}$ increases in odometer order. Let <u>left modal matrix</u> W have as columns the vectors Ev^k_{ij} followed by the vectors $Av^k_{\infty j}$ ordered as above. Then change bases in the domain (using V) and codomain (using W) of $(\lambda E-A)$ according to

$$W^{-1}(\lambda E-A)V = \begin{vmatrix} \lambda I-J & 0 \\ 0 & \lambda N-I \end{vmatrix} , \qquad (2.5)$$

where J is in Jordan form, and N consists of η nilpotent Jordan blocks each of size α_i. N describes the relative structure of (E,A) at ∞ and it is nilpotent with index ν, i.e. $N^{\nu-1} \neq 0$, $N^{\nu} = 0$. Note that dim $N(E)$ = dim $N(N)$. $(\lambda I-J) \oplus (\lambda N-I)$ is the <u>Weierstrass Form</u> of the pencil $(\lambda E-A)$.

If $E=I$, then $W=V$ and the above construction reverts to the transformation to Jordan form J of a single matrix A.

It is well known [12,6] that the rank k, $k > 1$, eigenvectors at ∞ give rise to impulsive behavior in (1.1) and following [19] we call such vectors <u>impulsive directions</u> of (1.1). Eigenvectors of rank 1 at ∞, on the other hand, do not give rise to impulsive behavior, and following [19] we call such directions <u>nonimpulsive directions at ∞</u>. Thus the trivial (i.e. length one) eigenvector chains at ∞ do not give rise to impulsive behavior.

The method of [6] for removing the impulsive behavior of (1.1) depends on proportional state feedback to reduce all eigenvector chains at ∞ to length one. In this process, $\rho-\eta$ new eigenvalues are introduced into the finite plane.

3. DEFINITIONS OF REACHABILITY AND CONTROLLABILITY

Apply the transformation (V,W) of section 2 to

$$P(s) = [sE-A \quad G] \qquad (3.1)$$

to obtain

$$[W^{-1}(sE-A)V \quad W^{-1}B] = \begin{vmatrix} sI-J & 0 & B \\ 0 & sN-I & D \end{vmatrix} , \qquad (3.2)$$

which corresponds to a decomposition of (1.1) into <u>slow</u> and <u>fast</u> subsystems [6,8,10,11]

$$\dot{x}_s = Jx_s + Bu \qquad (3.3a)$$

$$N\dot{x}_f = x_f + Du , \qquad (3.3b)$$

where $V^{-1}x = [x_s^T \quad x_f^T]^T$. Note that $x_s \in R^{n-\rho}$, $x_f \in R^{\rho}$.

Several definitions have been advanced for controllability in descriptor systems. Different definitions arise because of different ways of dealing with the trivial chains at ∞. See [22] which has resolved much of the confusion by a time-domain approach. In this section we shall present some relations which will help further clarify the notion(s) of controllability for descriptor systems. First we present two definitions.

Define (1.1) to be <u>reachable</u> (<u>reachable at ∞</u>) if for all x_o, $x_1 \in R^n$ ($x_o, x_1 \in H_N$) there exists an admissible control $u(t)$ on a finite nonzero interval $0 < t < T$ so that solution $x(t)$ to (1.1) satisfies $x(0) = x_o$, $x(T) = x_1$. This notion corresponds to C-controllability in [10]. See also [6,13,15,16].

Reachability at ∞ plus reachability in the conventional sense of state subsystem (3.3a) is equivalent to reachability of (1.1) as defined above [8].

Define (1.1) to be <u>modally controllable</u> (<u>modally controllable at ∞</u>) if all modes (all impulsive modes) can be excited from zero initial conditions using an input containing no component at the modal frequency (no impulses). This is the definition of [4,12,14]. We shall often omit the qualifier 'modally'.

Controllability at ∞ plus controllability in the conventional sense of state subsystem (3.3a) is equivalent to controllability of (1.1) [12].

Reachability and controllability in the conventional state space sense of finite subsystem (3.3a) are equivalent, and are equivalent to the full rank of P(s) for all finite s, or equivalently to the condition

$$\dim(\text{span}\{q \in R^{n-\rho} \mid q^T (sI-J)^{-1}B = 0\}) = 0 \ . \tag{3.4}$$

The next two theorems help clarify the notions of reachability and controllability at ∞. They relate several previously known results and present some new ones. Compare with [22] which contains portions of these results.

Theorem 3.1

The following conditions are equivalent.

a. (1.1) is reachable at ∞.

b. The rows of D corresponding to the bottom rows of all Jordan blocks of N are linearly independent.

c. rank $[D \;\; ND \;\; \ldots \;\; N^{\nu-1}D] = \rho$. [8,13]

d. $R(N) + R(D) = R^n$, where $R(\cdot)$ represents range [6].

e. $q^T(SN-I)^{-1}D = 0$ for constant q implies that q=0, i.e. $\dim(\text{span}\{q \in R^\rho \mid q^T(sN-I)^{-1}D=0\}) = 0$. (cf [15]).

f. The matrix [E-sA G] (or equivalently [N-sI D]) has full rank at s=0. [8,17].

proof:

These results are all well known except for b. and e., for which the proofs follow.

e. Note that $q^T(sN-I)^{-1}D = -q^T(I+sN + \ldots + s^{\nu-1}N^{\nu-1})\,D = 0$ for all s if and only if all coefficient matrices are zero, which is equivalent to

$$q^T[D \;\; ND \;\; \ldots \;\; N^{\nu-1}D] = 0$$

b. Due to the structure of N, for [N D] to have full rank it is necessary for the rows of D corresponding to all zero rows of N to be linearly independent. ●

Following [14] and in contrast to [17], define the infinite input decoupling zeros of (1.1) as the zeros at ∞ of P(s) [18]. i.e. as the zeros at s=0 of P(1/s).

Theorem 3.2

The following conditions are equivalent.

a. (1.1) is controllable at ∞.

b. The rows of D corresponding to the bottom rows of the nontrivial Jordan blocks of N are linearly independent [19].

c. rank$[ND \;\; \ldots \;\; N^{\nu-1}D] = \rho-\eta$. [cf 13].

d. $R(N) + R(D) + N(N) = R^\rho$ [6].

e. $\dim(\text{span}\{q \in R^\rho \mid q^T N(SN-I)^{-1}D=0\}) = \eta$.

f. $P(\frac{1}{s})$ (or equivalently $[\frac{1}{s}N-I \;\; D]$) has full rank at s=0.

g. $\begin{vmatrix} \bar{E} & 0 \\ -\tilde{A} & \tilde{G} \end{vmatrix}$ has full rank n, where a constant nonsingular transformation has been applied on the left of P(s) to obtain

$$\begin{vmatrix} s\bar{E}-\bar{A} & \bar{G} \\ -\tilde{A} & \tilde{G} \end{vmatrix} ,$$

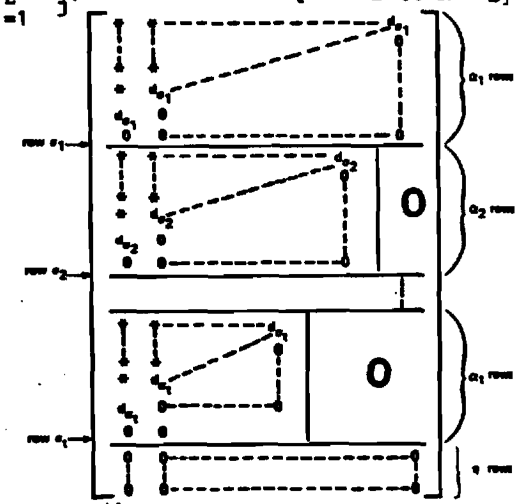
where $\bar{E}$ has full rank [12,19].

proof:

All these results are well known except c. and e., which are easily proved as follows.

c. Let $\sigma_i = \sum_{j=1}^{i} \alpha_j$, and note that $[ND \;\; N^2D \;..\; N^{\nu-1}D]$

=



where d_i is the i^{th} row of D, * represents an arbitrary element, and the blocks denoted by oversize 0's may be absent (e.g. if $\alpha_1 = \alpha_2$). There are $\rho - \sigma_t = \eta$ zero rows; and for the remainder of the matrix to have full rank it is necessary and sufficient that rows σ_i for $i = 1,\ldots,t$ of D be linearly independent, which is condition b.

e. As in the proof of Theorem 3.1 e., it follows that $q^T N(sN-I)^{-1}D = 0$ if and only if $q^T[ND \ldots N^{\nu-1}D] = 0$. ●

Note that conditions f. of the theorems show that reachability at ∞ is equivalent to the absence of infinite input decoupling zeros in the sense of Rosenbrock [17], while controllability at ∞ is equivalent to the absence of ∞ decoupling zeros in the sense of [14]. Conditions d. show on the other hand that controllability at ∞ is simply reachability at ∞ modulo N(E) [6]. Thus reachability implies controllability.

These results also hold with minor modifications for discrete descriptor systems. It is of interest to compare the relation between reachability and controllability at s=∞ for descriptor systems with the relation between nventional reachability and controllability r discrete state systems. The latter, of urse, depends on the eigenstructure at $s=0$.

In section 4 we present some results on en-loop control of discrete descriptor stems. The important property in that ection is reachability. In section 5 we eneralize a result of Moore [20] on closed-op eigenstructure assignability. The portant property there is controllability.

4. REACHABILITY AND OPEN-LOOP CONTROL

This section summarizes some results in 38], and is presented here in juxtaposition ith the results on controllability in the ext section in order to obtain a complete omparison between the properties of reachbility and controllability for descriptor ystems. To simplify matters the discrete escriptor formulation is treated. The proofs an be found in [25].

Consider the discrete version of (1.1)

$$Ex_{k+1} = Ax_k + Gu_k \; ; \quad k = 0,\ 1,\ \ldots,\ N-1 \quad (4.1)$$

here $u_k \in R^m$, $x_k \in R^n$, and integer N specfies the time interval of interest.

Since (4.1) is regular, we can write the nique Laurent expansion for the resolvent atrix as

$$(zE-A)^{-1} = z^{-1} \sum_{k=-\nu}^{\infty} \phi_k z^{-k} , \qquad (4.2)$$

where ν is the index of nilpotency of the pencil $(zE-A)$. We call ϕ_k the fundamental matrix for (4.1). If $E=I$ then $\phi_k = A^k$ and $\nu = 0$.

The importance of the descriptor fundamental matrix has been discussed, but is not generally realized. In [26] it is shown how to compute $\{\phi_k\}$ given ϕ_0 and ϕ_{-1}, and in [24] it is shown how to compute $\{\phi_k\}$ from E and A by using the Drazin inverse. It is known [15,26] that $\phi_0 E$ is the projection on H_I along H_N, and that $-\phi_{-1}A$ is the projection on H_N along H_I. The descriptor open-loop control problem is solved here in terms of ϕ_k.

Given x_1, $x_2 \in R^n$, define the pair (x_1,x_2) to be reachable if there exists a control $\bar{u}_{0,N}$ for some $N > 0$ such that $\bar{x}_{0,N+1}$ is a solution to (4.1) with $x_0 = x_1$, $x_N = x_2$. We shall loosely speak of (x_0,x_N) as reachable. Define system (4.1) to be reachable if all pairs (x_0,x_N) are reachable for some $N > 0$.

Since it does not guarantee the coupling between the input and the trivial modes at infinity, it is clear that controllability is not sufficient to solve the open-loop control problem.

Theorem 4.1

Let (4.1) be regular and define descriptor reachability matrix

$$U_N = \begin{vmatrix} \phi_0 G & \phi_1 G & \cdots & \phi_{N-1} G \\ \phi_{-N} G & \cdots & \phi_{-2} G & \phi_{-1} G \end{vmatrix} . \qquad (4.3)$$

Then over any interval $[0,N]$, the control sequence and the initial and final values of the descriptor variable are related by

$$\begin{vmatrix} \phi_0 E & -\phi_{N-1} A \\ \phi_{-N} E & -\phi_{-1} A \end{vmatrix} \begin{vmatrix} x_N \\ x_0 \end{vmatrix} = U_N \, \bar{u}_{0,N} . \qquad (4.4)$$

Furthermore, reachability for (4.1) can be studied in terms of (4.4). ●

In (4.3), $\phi_k = 0$ for $k < -\nu$. For ease of notation this is not explicitly indicated.

From this theorem there follow several results whose proofs are quite trivial. Superscripts "-1" and "+" denote inverse image of a linear operator and Moore-Penrose Matrix inverse respectively. The first result provides a reachability test.

Corollary 4.2

A regular system (4.1) is reachable if and only if, for some $N > 0$,

$$R\left(\begin{vmatrix} \phi_0 E & \phi_{N-1} A \\ \phi_{-N} E & \phi_{-1} A \end{vmatrix}\right) \quad R(U_N) . \qquad (4.5)●$$

Next, we characterize the reachable subspace of (4.1).

Corollary 4.3

Let (4.1) be regular. Then (x_0,x_N) is reachable if and only if

$$\begin{vmatrix} x_N \\ x_o \end{vmatrix} \in \begin{vmatrix} \phi_0 E & -\phi_{N-1} A \\ \phi_{-N} E & -\phi_{-1} A \end{vmatrix}^{-1} R(U_N) . \qquad (4.6)●$$

If $E=I$ then with $N=n$ (4.4) becomes

$$\begin{vmatrix} I & -A^n \\ 0 & 0 \end{vmatrix} \begin{vmatrix} x_n \\ x_o \end{vmatrix} \equiv \begin{vmatrix} G & AG & \cdots & A^{n-1}G \\ 0 & 0 & \cdots & 0 \end{vmatrix} \bar{u}_{0,n}$$

(4.7)

which is equivalent to $(x_n - A^n x_o) \in R[G \;\; AG \;\; \ldots \;\; A^{n-1}G]$, the familiar state system result. Note that the usual definitions of reachability and controllability for discrete state systems both derive from our single generalized definition for reachability of the pair (x_o,x_n). Hence, in the usual terminology, a state system is reachable if $x_n \in R[G \;\; AG \; \ldots \; A^{n-1}G]$ for all $x_n \in R^n$, and controllable if $A^n x_o \in R[G \;\; AG \ldots A^{n-1}G]$ for all $x_o \in R^n$.

The open-loop control problem for descriptor systems is solved next.

Corollary 4.4

The minimum-norm control which makes the descriptor variable of the regular system (4.1) take on prescribed values x_0 and x_N exists if and only if (4.6) holds, and then it is given by

$$\bar{u}_{0,N} = U_N^+ \begin{vmatrix} \phi_0 E & -\phi_{N-1} A \\ \phi_{-N} E & -\phi_{-1} A \end{vmatrix} \begin{vmatrix} x_N \\ x_0 \end{vmatrix} . \qquad (4.8)\bullet$$

This reduces to the well-known state space result in the case E=I.

5. CONTROLLABILITY AND EIGENSTRUCTURE ASSIGNMENT

It is well-known that modal controllability and eigenvalue assignability are equivalent for descriptor systems [21]. In this section we generalize a result by Moore [20] to the case of descriptor systems.

Current approaches to descriptor system pole assignment employ a "fast feedback" to eliminate impulsive behavior, and then a second conventional state feedback on the resulting state subsystem to place the poles as desired [6,21,23]. In this section we show how to select a single feedback which directly assigns closed-loop poles and eigenvectors. Only the case of distinct closed-loop poles is treated. Theorem 5.1 is a direct and simple generalization of Moore's result.

A key factor here is that although $|\lambda E-A| \neq 0$, it is not guaranteed for all F that $|\lambda E-A-GF| \neq 0$. Our feedback design method must guarantee this.

Let $[L^T(\lambda) \;\; M^T(\lambda)]^T$ be a rational basis for the null space of $P(\lambda)$ so that

$$[\lambda E-A \quad G] \begin{vmatrix} L(\lambda) \\ M(\lambda) \end{vmatrix} = 0 . \qquad (5.1)$$

Then we have the following result which shows how to select feedback F to yield a desired closed-loop eigenstructure.

Theorem 5.1

Let $\{\lambda_i\}_1^{n-\eta}$ be a self-conjugate set of distinct finite complex numbers, where η = dim N(E). There exists a real matrix F such that $(A + GF)v_i = \lambda_i E v_i$, $i = 1,\ldots,n-\eta$ and $|\lambda E-A-FG| \neq 0$ if and only if

1. Vectors Ev_i, $i = 1,\ldots,n-\eta$ are linearly independent in C^n,
2. $v_i = v_j^*$ when $\lambda_i = \lambda_j^*$, where * denotes complex conjugation,
3. $v_i \in R(L(\lambda_i))$.

proof:

Sufficiency

If 3. holds then $v_i = L(\lambda_i)k_i$ for some vector k_i. Then by (5.1)

$$(\lambda_i E-A)v_i + GM(\lambda_i)k_i = 0 .$$

Let $w_i = -M(\lambda_i)k_i$ and define F by

$$F[v_1 \; v_2 \; \ldots \; v_{n-\eta}] = [w_1 \; w_2 \; \ldots \; w_{n-\eta}] . \qquad (5.2)$$

Then

$$0 = (\lambda_i E-A)v_i - GFv_i$$
$$= (\lambda_i E-(A+GF))v_i = 0 , \quad \text{for } i = 1,\ldots,n-\eta .$$

Therefore the finite and infinite eigenspaces with feedback are given by

$$H_{IF} = \mathrm{span}\{v_i\}, \; H_{NF} = N(E) .$$

Now $\{Ev_i\}$ linearly independent implies the restriction $E|H_{IF}$ is one to one; so $(\lambda E-(A+GF))$ is regular [7].

To show (5.2) has a solution, write it as FV=W, which has a solution if and only if $N(V) \subset N(W)$. But $\{Ev_i\}$ linearly independent implies $\{v_i\}$ linearly independent so that N(V) = 0. To show solution F can be chosen as real, use the construction in [20].

Necessity

Let

$$(A+GF)v_i = \lambda_i E v_i \quad \text{for } i = 1,\ldots,n-\eta , \qquad (5.3)$$

with F real. Then $H_{IF} = \mathrm{span}\{v_i\}$. By [7], $(\lambda E-(A+GF))$ regular implies $\{Ev_i\}$ linearly independent. F real implies condition 2. Finally, (5.3) implies

$$[\lambda_i E-A \quad G] \begin{vmatrix} v_i \\ -Fv_i \end{vmatrix} = 0 ,$$

which in turn implies 3. $\bullet$

This theorem implicitly contains the result that at most $n-\eta$ relative eigenvalues can be assigned to the finite closed-loop spectrum. After applying the feedback only η = dim N(E) eigenvalues remain in the infinite closed-loop spectrum; so only rank 1 eigenvectors at infinity remain. Accordingly, the feedback constructed by (5.2) eliminates the impulsive behavior of (1.1). Compare with [6,23].

6. CONCLUSIONS

We have shown that the distinct definitions of controllability for descriptor systems in the literature arise from different ways of dealing with the eigenvector chains of length one at infinity. Defining two properties, which we have called reachability and (modal) controllability, removes the confusion.

We discussed reachability, defining a descriptor reachability matrix and solving the generalized open-loop control problem. Then we discussed pole assignability, which depends on controllability, presenting a generalized version of a theorem by Moore on closed-loop assignment of eigenstructure.

The distinction between descriptor system reachability and controllability depends on the eigenstructure at infinity, much as the distinction between the conventional properties of reachability and controllability for discrete state systems depends on the eigenstructure at zero.

REFERENCES

[1] F. R. Gantmacher, The Theory of Matrices, New York: Chelsea, 1974.

[2] F. L. Lewis and K. Ozcaldiran, "The relative eigenstructure problem and descriptor systems," SIAM National Meeting, Denver, CO, June 1983.

[3] N. Karcanias and G. E. Hayton, "Generalized autonomous dynamical systems, algebraic duality and geometric theory," IFAC VIII Triennial World Congress, Kyoto, Japan, August 1981.

[4] P. M. Van Dooren, "The generalized eigenstructure problem in linear system theory," IEEE Trans. Automat. Contr., vol. AC-26, no. 1, pp. 111-129, Feb. 1981.

[5] G. C. Verghese and T. Kailath, "Eigenvector chains for finite and infinite zeros of rational matrices," Proc. 18th Conf. Dec. & Control, Ft. Lauderdale, FL, pp. 31-32, December 1979.

[6] D. Cobb, "Feedback and pole placement in descriptor variable systems," Int. J. Control, vol. 33, no. 6, pp. 1135-1146, 1981.

[7] K-T Wong, "The eigenvalue problem λTx + Sx," J. Diff. Equations, vol. 16, pp. 270-281, 1974.

[8] E. L. Yip and R. F. Sincovec, "Solvability, Controllability, and Observability of Continuous Descriptor Systems," IEEE Trans. Automat. Contr., pp. 702-707, June 1981.

[9] F. L. Lewis, "Inversion of descriptor systems," Proc. ACC, San Francisco, CA, pp. 1153-1158, June 1983.

[10] F. L. Lewis, "Descriptor systems: decomposition into forward and backward subsystems," IEEE Trans. Automat. Contr., vol. AC-29, no. 1, Jan. 1984.

[11] S. L. Campbell, C. D. Meyer, Jr., and N. J. Rose, "Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients," SIAM J. Appl. Math., vol. 31, no. 3, pp. 411-425, Nov. 1976.

[12] G. C. Verghese, B. C. Levy, and T. Kailath, "A generalized state-space for singular systems," IEEE Trans. Automat. Contr., vol. AC-26, no. 4, pp. 811-831, August 1981.

[13] L. Pandolfi, "Controllability and stabilization for linear systems of algebraic and differential equations," JOTA, vol. 30, no. 4, pp. 601-620, April 1980.

[14] G. C. Verghese and T. Kailath, "Impulsive behavior in dynamical systems: structure and significance," Proc. 4th Int. Symp. Math. Theory, Networks Syst., Delft, The Netherlands, pp. 162-168, July 1979.

[15] C. E. Langenhop, "Controllability and stabilization of regular singular linear systems with constant coefficients," Dept. of Mathematics, S. Illinois Univ., Dec. 6, 1979.

[16] E. N. Khasina, "Control of singular linear dynamic systems," Automation and Remote Control, vol. 43, no. 4, pp. 448-455, April 1982.

[17] H. H. Rosenbrock, "Structural properties of linear dynamical systems," Int. J. Control, vol. 20, no. 2, pp. 191-202, 1974.

[18] A. C. Pugh and P. A. Ratcliffe, "On the zeros and poles of a rational matrix," Int. J. Control, vol. 30, no. 2, pp. 213-226, 1979.

[19] G. C. Verghese, "Infinite-frequency behavior in generalized dynamical systems," Ph.D. Thesis, Dept. of Electrical Engineering, Stanford University, 1979.

[20] B. C. Moore, "On the flexibility offered by state feedback in multivariable systems beyond closed loop eigenvalue assignment," IEEE Trans. on Automatic Control, pp. 680-692, October 1976.

[21] R. Mukundan and W. Dayawansa, "Feedback of singular systems-proportional and derivative feedback of the state," Int. J. Systems Sci., vol. 14, no. 6, pp. 615-632, 1983.

[22] D. Cobb, "Descriptor variable systems and optimal state regulation," IEEE Trans. Automat. Contr., vol. AC-28, no. 5, pp. 601-611, May 1983.

[23] S. L. Campbell, Singular Systems of Differential Equations, vols. I and II Pitman, 1980.

[24] N. J. Rose, "The Laurent expansion of a generalized resolvent with some applications," SIAM J. Math. Analy., vol. 9, no. 4, pp. 751-758, Aug. 1978.

[25] F. L. Lewis, "Descriptor Systems: fundamental matrix, reachability and observability matrices, subspaces," submitted to 23rd CDC.

[26] C. E. Langenhop, "The Laurent expansion for a nearly singular matrix," Linear Algebra and its Applic., 4, pp. 329-340, 1971.

27TH MIDWEST SYMPOSIUM ON CIRCUITS AND SYSTEMS

Volume II

June 11-12, 1984
West Virginia University
Morgantown, West Virginia

Proceedings of 23rd Conference
on Decision and Control
Las Vegas, NV, December 1984

DESCRIPTOR SYSTEMS: FUNDAMENTAL MATRIX, REACHABILITY AND OBSERVABILITY MATRICES, SUBSPACES*

F. Lewis

School of Electrical Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332

Abstract

This paper uses the fundamental matrix of a regular discrete descriptor system to derive expressions for descriptor reachability and observability matrices. Reachable and unobservable subspaces and a subspace of admissible boundary conditions are defined. It is shown that the natural space for analyzing descriptor system properties seems to be R^{2n} (where n is the dimension of the system), not R^n as is the case for state-space systems. Solutions are provided for the descriptor open-loop control and estimation problems.

I. INTRODUCTION

There have been many approaches to analysis of descriptor or generalized state systems. These may basically be divided into time domain methods [1-6,15,22] and polynomial matrix/matrix pencil methods [7-9,19,20,23,25]. Various approaches have used the Drazin inverse [3,9,15,22], generalized eigenstructure [13,25,27-30], Weierstrass form [4,6,7,17,18, 24,25,27,31], numerical methods [6], and the concept of the output-zeroing problem [32]. The above approaches are not mutually exclusive and indeed their interdependencies are what make the study of descriptor systems so fascinating. These systems provide a focus which highlights some new relations between many different techniques.

In this paper we use a time-domain point of view and the descriptor fundamental matrix to define reachability and observability matrices. Our approach allows us to define reachable and unobservable subspaces in terms of the descriptor fundamental matrix. We are also able to solve the descriptor open-loop control and estimation problems.

A distinguishing feature of our work is that, in consonance with the noncausal or "nonoriented" nature of descriptor systems, we work with

$$\begin{vmatrix} x_0 \\ x_N \end{vmatrix} \in R^{2n}, \tag{1.1}$$

where 0, N represent initial and final times. This allows us to help clarify the duality between reachability and observability in the time domain (see also [17] which treats continuous systems). All of our results depend on the fundamental matrix; so no transformation to special form is required.

The focus here is on discrete systems, though the descriptor Cayley-Hamilton theorem [11] should allow a generalization to continuous systems.

*Supported by NSF Grant ECS-8204656.

At each step it is shown how the familiar state space results are recovered as a special case of the results presented herein.

II. FUNDAMENTAL MATRIX AND CAYLEY-HAMILTON THEOREM

Consider the linear time invariant system over the real numbers R

$$Ex_{k+1} = Fx_k + Gu_k \tag{2.1a}$$

$$y_k = Hx_k \; ; \; k = 0, 1, \ldots, N-1 \tag{2.1b}$$

where $u_k \in R^m$, $x_k \in R^n$, $y_k \in R^p$, and N specifies the time interval of interest. If E is singular then x_k should not be considered the state of (2.1), and following Luenberger [1] we call x_k the descriptor variable. Note that if E is nonsingular it is possible to solve for x_k by forward iteration given x_0. On the other hand, if F is nonsingular it is possible to solve for x_k by backward iteration given x_N. In general, however, it is not possible to solve (2.1) by simple iteration in one direction [2-6].

We assume throughout that (2.1) is regular, i.e. $|zE-F| \neq 0$ [1,2,7,8]. In this case we can write the unique Laurent expansion for the resolvent matrix for large values of z as

$$(zE-F)^{-1} = z^{-1} \sum_{k=-\mu}^{\infty} \phi_k z^{-k} , \tag{2.2}$$

where μ is the index of nilpotency of the pencil $(zE-F)$ [9-11]. We call $\{\phi_k\}$ the <u>fundamental matrix</u> for (2.1). Note that ϕ_k satisfies the well-known equalities

$$E\phi_k - F\phi_{k-1} = \delta_{ok} I , \tag{2.3a}$$

$$\phi_k E - \phi_{k-1} F = \delta_{ok} I , \tag{2.3b}$$

where δ_{ok} is the Kronecker delta [10,12].

The importance of the descriptor fundamental matrix has been discussed, but is not generally realized. In [12] it is shown how to compute $\{\phi_k\}$ given ϕ_0 and ϕ_{-1}, and in [9] it is shown how to compute $\{\phi_k\}$ from E and F by using the Drazin inverse. It is known [10,12] that $\phi_0 E$ is the projection on H_I along H_N, and that $-\phi_{-1}F$ is the projection on H_N along H_I, where the finite and infinite eigenspaces H_I and H_N are defined in [12-14]. H_I may also be interpreted as the subspace of admissible initial conditions with zero input [1,2,13].

In this paper we focus on the properties and uses of the descriptor fundamental matrix. The first result is the following.

Theorem 2.1

Let

$$\Delta(z) = |zE-F| = p_0 z^n - p_1 z^{n-1} - \dots - p_n \,. \quad (2.4)$$

Then

$$p_0\phi_k - p_1\phi_{k-1} - \dots - p_n\phi_{k-n} = 0 \qquad (2.5)$$

for $k > n$ and $k < -1$.

proof: see [11]. ●

If $E=I$, then $\mu = 0$ and $\phi_k = F^k$; so that (2.5) with $k=n$ becomes simply $\Delta(F) = 0$. We therefore call Theorem 2.1 the descriptor Cayley-Hamilton theorem.

We shall require the following notation. Write (2.1) in expanded form (c.f. [1,2]) as

$$\begin{vmatrix} E & -F & 0 & \cdots & 0 & 0 \\ 0 & E & -F & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & & & \ddots & -F & 0 \\ 0 & \cdots & \cdots & \cdots & E & -F \end{vmatrix} \begin{vmatrix} x_N \\ \vdots \\ x_1 \\ x_0 \end{vmatrix} = \begin{vmatrix} G & & 0 \\ 0 & \ddots & G \end{vmatrix} \begin{vmatrix} u_{N-1} \\ \vdots \\ u_1 \\ u_0 \end{vmatrix} \qquad (2.6a)$$

$$\begin{vmatrix} y_{N-1} \\ \vdots \\ y_1 \\ y_0 \end{vmatrix} = \begin{vmatrix} H & & 0 \\ 0 & \ddots & H \end{vmatrix} \begin{vmatrix} x_{N-1} \\ \vdots \\ x_1 \\ x_0 \end{vmatrix} , \qquad (2.6b)$$

or by appropriate definition of the coefficient matrices A_N, B_N, C_N and the input, descriptor, and output sequence vectors $\bar{u}_{o,N}$, $\bar{x}_{o,N}$, $\bar{y}_{o,N}$ as

$$A_N \,\bar{x}_{o,N+1} = B_N \,\bar{u}_{o,N} \qquad (2.7a)$$

$$\bar{y}_{o,N} = C_N \,\bar{x}_{o,N} \,. \qquad (2.7b)$$

Define also the auxiliary matrix

$$a_N = \begin{vmatrix} -F & 0 & \cdots & 0 & 0 \\ E & -F & & \vdots & \vdots \\ 0 & E & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \cdots & E & -F \\ 0 & 0 & \cdots & 0 & E \end{vmatrix}_{nN \times n(N-1)} \qquad (2.8)$$

which is just A_N with the first and last block columns deleted.

The next results will subsequently be required.

Lemma 2.2

Suppose (2.1) is regular. Then a right inverse of A_N for any $N > 0$ is given by

$$A_N^r = \begin{vmatrix} \phi_0 & \phi_1 & \phi_2 & \cdots & \phi_{N-1} \\ \phi_{-1} & \phi_0 & \phi_1 & \cdots & \phi_{N-2} \\ \vdots & & \ddots & \ddots & \vdots \\ & & & \ddots & \phi_0 \\ \phi_{-N} & \cdots & \cdots & \cdots & \phi_{-1} \end{vmatrix} . \qquad (2.9)$$

proof: Show $A_N A_N^r = I$ by using (2.3a). ●

Lemma 2.3

Suppose (2.1) is regular. Then a left inverse of a_N for any $N > 0$ is given by

$$a_N^\ell = \begin{vmatrix} \phi_{-1} & \phi_0 & \phi_1 & \cdots & \phi_{N-2} \\ \phi_{-2} & \phi_{-1} & \phi_0 & \cdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \phi_{-N+1} & \cdots & & \phi_{-1} & \phi_0 \end{vmatrix} . \qquad (2.10)$$

proof: Show $a_N^\ell a_N = I$ by using (2.3b). ●

It is well known [1,2] that regularity of (2.1) is equivalent to the full rank of A_N and a_N, and hence of A_N^r and a_N^ℓ, for all $N > 1$.

III. REACHABILITY

Given $z_1, z_2 \in R^n$, define the pair (z_1,z_2) to be reachable if there exists a control $\bar{u}_{0,N}$ for some $N > 0$ such that $\bar{x}_{0,N+1}$ is a solution to (2.1a) with $x_0 = z_1$, $x_N = z_2$. We will loosely speak of (x_0,x_N) as reachable. Define system (2.1) to be reachable if all pairs (x_0,x_N) are reachable. This is consistent with definitions in [4,10,15].

It is well known [4] that (2.1) is reachable if and only if

$$\mathrm{rank}[zE-F \quad G] = n \text{ all } z \,, \qquad (3.1a)$$

and

$$\mathrm{rank}[E \quad G] = n \,. \qquad (3.1b)$$

The next result presents one possible condition for reachability in terms of the fundamental matrix.

Theorem 3.1

If (2.1) is regular, it is reachable if and only if

$$\mathrm{rank}[\phi_{-\mu}G \;\dots\; \phi_{-1}G \quad \phi_0 G \;\dots\; \phi_{n-1}G] = n \,. \quad (3.2)$$

proof:

Reachability is equivalent [10] to the condition

$$q^T(zE-F)^{-1}G = 0 \text{ for } q \in R^n, \text{ all } z \text{ implies } q=0 \,. \qquad (3.3)$$

By using (2.2) and (2.5), the theorem follows readily. ●

If $E=I$, then condition (3.2) becomes

$$\mathrm{rank}\,[G \quad FG \;\dots\; F^{n-1}G] = n \,.$$

Theorem (3.1) is not useful for our objectives, which include the computation of the open-loop control required to make the solution to (2.1) have desired values of x_0 and x_N. Accordingly the next result is presented.

Theorem 3.2

Let (2.1) be regular and define descriptor reachability matrix

$$U_N = \begin{vmatrix} \phi_0 G & \phi_1 G & \cdots & \phi_{N-1}G \\ \phi_{-N}G & \cdots & \phi_{-2}G & \phi_{-1}G \end{vmatrix} . \qquad (3.4)$$

Then over any interval $[0,N]$, the control sequence and the initial and final values of the descriptor variable are related by

$$\begin{vmatrix} \phi_0 E & -\phi_{N-1}F \\ \phi_{-N}E & -\phi_{-1}F \end{vmatrix} \begin{vmatrix} x_N \\ x_0 \end{vmatrix} = U_N \, \bar{u}_{0,N} \, . \qquad (3.5)$$

:thermore, reachability for (2.1) can be studied terms of (3.5).

ɔof:

Rewrite (2.6a)/(2.7a)

$$\begin{vmatrix} -Ex_N \\ 0 \\ \vdots \\ 0 \\ Fx_0 \end{vmatrix} = [A_N \; B_N] \begin{vmatrix} \bar{x}_{1,N} \\ -\bar{u}_{0,N} \end{vmatrix} \, . \qquad (3.6)$$

emultiply both sides of (3.6) by A_N^r as given by .9) to obtain

$$\begin{vmatrix} -\phi_0 E & \phi_{N-1}F \\ -\phi_{-1}E & \phi_{N-2}F \\ \vdots & \vdots \\ -\phi_{-N+1}E & \phi_0 F \\ -\phi_{-N}E & \phi_{-1}F \end{vmatrix} \begin{vmatrix} x_N \\ x_0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \phi_0 G & \phi_1 G & \cdots & \phi_{N-1}G \\ & \phi_{-1}G & \phi_0 G & \cdots & \phi_{N-2}G \\ I & \vdots & & & \vdots \\ & \phi_{-N+1}G & & & \phi_0 G \\ 0 & \phi_{-N}G & \cdots & \phi_{-2}G & \phi_{-1}G \end{vmatrix} \begin{vmatrix} \bar{x}_{1,N} \\ -\bar{u}_{0,N} \end{vmatrix} \, . \qquad (3.7)$$

nce A_N^r has full column rank (3.6) has a solution and only if (3.7) does. The form of (3.7) guartees that we can solve for $\bar{x}_{1,N}$ given any x_0, $\bar{u}_{0,N}$ which are otherwise consistent with 3.7). Therefore (3.6) is equivalent to (3.5) •

In (3.4), $\phi_k = 0$ for $k < -\mu$. For ease of notaon this is not mplicitly indicated.

From this theorem there follow several results se proofs are quite trivial. $R(\cdot)$ denotes range linear operator, and superscripts "-1" and "+" ote inverse image of a linear operator and Moorerose Matrix inverse respectively. The first ult provides another reachability test.

ollary 3.3

A regular system (2.1) is reachable if and only for some $N > 0$,

$$R\left(\begin{vmatrix} \phi_0 E & \phi_{N-1}F \\ \phi_{-N}E & \phi_{-1}F \end{vmatrix}\right) \subset R(U_N) \, . \qquad (3.8)\bullet$$

Next, we characterize the reachable subspace of 1).

ollary 3.4

Let (2.1) be regular. Then (x_0, x_N) is reache if and only if

$$\begin{vmatrix} x_N \\ x_0 \end{vmatrix} \in \begin{vmatrix} \phi_0 E & -\phi_{N-1}F \\ \phi_{-N}E & -\phi_{-1}F \end{vmatrix}^{-1} R(U_N) \, . \qquad (3.9)\bullet$$

Equation (3.9) should be compared to the equivalent characterization by deflating subspaces in [25] and by Weierstrass form in [4].

If $E=I$ then with $N=n$ (3.5) becomes

$$\begin{vmatrix} I & -F^n \\ 0 & 0 \end{vmatrix} \begin{vmatrix} x_n \\ x_0 \end{vmatrix} = \begin{vmatrix} G & FG & \cdots & F^{n-1}G \\ 0 & 0 & \cdots & 0 \end{vmatrix} \bar{u}_{0,n} \qquad (3.10)$$

which is equivalent to $(x_n - F^n x_0) \in R[G \;\; FG \;\; \ldots \;\; F^{n-1}G]$, the familiar state system result. Note that the usual definitions of reachability and controllability for discrete state systems both derive from our single definition for reachability of the pair (x_0, x_n). Hence, in the usual terminology, a state system is reachable if $x_n \in R[G \;\; FG \;\; \ldots \;\; F^{n-1}G]$ for all $x_n \in R^n$, and controllable if $F^n x_0 \in R[G \;\; FG \;\; \ldots \;\; F^{n-1}G]$ for all $x_0 \in R^n$.

The open-loop control problem for descriptor systems is solved next.

Corollary 3.5

The minimum-norm control which makes the descriptor variable of the regular system (2.1) take on prescribed values x_0 and x_N exists if and only if (3.9) holds, and then it is given by

$$\bar{u}_{0,N} = U_N^+ \begin{vmatrix} \phi_0 E & -\phi_{N-1}F \\ \phi_{-N}E & -\phi_{-1}F \end{vmatrix} \begin{vmatrix} x_N \\ x_0 \end{vmatrix} \, . \qquad (3.11)\bullet$$

This reduces to the well-known state space result in the case $E=I$.

It is worthwhile to compare the notions of reachability and solvability [1,2,16]. (2.1) is solvable if a solution $\bar{x}_{0,N+1}$ exists for all $\bar{u}_{0,N}$. In general this occurs if and only if $R(B_N) \subset R(A_N)$, or equivalently rank $[zE-F]$ = rank$[zE-F \;\; G]$ for almost every z [16]. Compare this condition to (3.1). Regularity implies solvability. (In [1,2], solvability and regularity are defined to be equivalent, though in fact they should be thought of as distinct properties. See [16].) Solvability is defined in terms of (2.7a) while reachability is defined in terms of (3.7), or equivalently (3.5).

IV. OBSERVABILITY

Observability for descriptor systems seems to be a more difficult problem conceptually than reachability, as is attested to by the dearth of references on the subject. We deal with a form of observability which gives dual results in the time domain to those of section III, and which reduces if $E=I$ to the state space results. Our definition is similar to that in [17,20] and should be contrasted to the definition in [4].

Given $z_1, z_2 \in R^n$, define the pair (z_1, z_2) to be observable if for $\bar{u}_{0,N+1} = 0$ and some $N > 0$, knowledge of the output $\bar{y}_{1,N+1}$ resulting when $x_0 = z_1$ and $x_{N+1} = z_2$ is sufficient to uniquely determine Fx_0, Ex_{N+1}. We shall loosely speak of (x_0, x_{N+1}) as observable. Define system (2.1) to be observable if all pairs (x_0, x_{N+1}) are observable. It will become clear that the choice of final time (i.e. N+1 not N) will result in a duality with the results of section III. (Note: the pair (x_0, x_{N+1}) must be an admissible set of boundary values for (2.1).)

Theorem 4.1

Let (2.1) be regular and define <u>descriptor observability matrix</u>

$$V_N = \begin{vmatrix} H\phi_{-1} & H\phi_{N-1} \\ H\phi_{-2} & \vdots \\ \vdots & H\phi_1 \\ H\phi_{-N} & H\phi_0 \end{vmatrix} . \tag{4.1}$$

Then over any interval $[0, N+1]$, the output and the initial and final values of the descriptor variable are related by

$$\begin{vmatrix} F\phi_{-1} & F\phi_{N-1} \\ E\phi_{-N} & E\phi_0 \\ \hline \multicolumn{2}{c}{V_N} \end{vmatrix} \begin{vmatrix} w_N \\ w_0 \end{vmatrix} = \begin{vmatrix} Ex_{N+1} \\ Fx_0 \\ \hline \bar{y}_{1,N+1} \end{vmatrix} \tag{4.2a}$$

$$\begin{vmatrix} H & 0 \\ 0 & H \end{vmatrix} \begin{vmatrix} x_{N+1} \\ x_0 \end{vmatrix} = \begin{vmatrix} y_{N+1} \\ y_0 \end{vmatrix} , \tag{4.2b}$$

where w_0, w_N are intermediate variables. Furthermore, the observability of (2.1) can be studied in terms of (4.2).

<u>proof</u>: Let $\bar{u}_{0,N+1} = 0$ and write (2.6)/(2.7) as

$$\begin{vmatrix} A_{N+1} \\ C_{N+2} \end{vmatrix} \bar{x}_{0,N+2} = \begin{vmatrix} 0 \\ \bar{y}_{0,N+2} \end{vmatrix} ,$$

which can be rewritten as

$$\begin{vmatrix} a_{N+1} \\ \hline C_N \end{vmatrix} \bar{x}_{1,N+1} = \begin{vmatrix} -Ex_{N+1} \\ 0 \\ 0 \\ Fx_0 \\ \hline \bar{y}_{1,N+1} \end{vmatrix} \tag{4.3a}$$

and

$$\begin{vmatrix} y_{N+1} \\ y_0 \end{vmatrix} = \begin{vmatrix} H & 0 \\ 0 & H \end{vmatrix} \begin{vmatrix} x_{N+1} \\ x_0 \end{vmatrix} . \tag{4.3b}$$

Define intermediate variable $\bar{w}_{0,N+1}$ by $\bar{x}_{1,N+1} = a^{\ell}_{N+1}\bar{w}_{0,N+1}$ and write (4.3a) as

$$\begin{vmatrix} L \\ \hline M \end{vmatrix} \bar{w}_{0,N+1} \triangleq \left|\begin{array}{c|ccc|c} -F\phi_{-1} & -F\phi_0 & \cdots & -F\phi_{N-2} & -F\phi_{N-1} \\ \hline 0 & & I & & 0 \\ \hline E\phi_{-N} & E\phi_{-N+1} & \cdots & E\phi_{-1} & E\phi_0 \\ \hline \hline H\phi_{-1} & H\phi_0 & \cdots & H\phi_{N-2} & H\phi_{N-1} \\ H\phi_{-2} & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & H\phi_1 \\ H\phi_{-N} & H\phi_{-N+1} & \cdots & H\phi_{-1} & H\phi_0 \end{array}\right|$$

$$\bar{w}_{0,N+1} = \begin{vmatrix} -Ex_{N+1} \\ \hline 0 \\ \hline Fx_0 \\ \hline \hline \bar{y}_{1,N+1} \end{vmatrix} \tag{4.4}$$

Now, given $\bar{y}_{1,N+1}$, (4.3a) has a unique solution with respect to Fx_0, Ex_{N+1} (i.e. possibly different $\bar{x}_{1,N+1}$ give rise to the same Fx_0, Ex_{N+1}) if and only if $N(C_N) \subset N(a^{\ell}_{N+1})$. Since a_{N+1} has full row rank, this is equivalent to $N(M) \subset N(L)$. Thus (4.3a) and (4.4) are equivalent with respect to uniqueness of solution. Due to the structure of L, $w_{1,N} = 0$ and (4.4) is effectively equivalent to (4.2a). ●

At this point the next results are immediate.

<u>Corollary 4.2</u>

A regular system (2.1) is observable if and only if, for some $N > 0$,

$$N(V_N) \subset N \begin{vmatrix} F\phi_{-1} & F\phi_{N-1} \\ E\phi_{-N} & E\phi_0 \end{vmatrix} . \tag{4.5}$$

<u>proof</u>:

For a given $\bar{y}_{1,N+1}$, the solution to (4.2a) is unique with respect to Fx_0, Ex_{N+1} if and only if (4.5) obtains. ●

The unobservable subspace is characterized next. Superscript "$\perp$" denotes orthogonal complement of a subspace.

<u>Corollary 4.3</u>

Let (2.1) be regular. Then an admissible pair (x_0, x_{N+1}) makes a zero contribution to the output iff

$$\begin{vmatrix} Ex_{N+1} \\ Fx_0 \end{vmatrix} \in \left(\begin{vmatrix} F\phi_{-1} & F\phi_{N-1} \\ E\phi_{-N} & E\phi_0 \end{vmatrix} N(V_N) \right) . \tag{4.6}$$

<u>proof</u>: Any $\begin{vmatrix} w_N \\ w_0 \end{vmatrix} \in N(V_N)$ results in $\bar{y}_{1,N+1} = 0$.

Such "unobservable intermediate variables" contribute a component to the solution given by

$$\begin{vmatrix} Ex_{N+1} \\ Fx_0 \end{vmatrix} = \begin{vmatrix} F\phi_{-1} & F\phi_{N-1} \\ E\phi_{-N} & E\phi_0 \end{vmatrix} \begin{vmatrix} w_N \\ w_0 \end{vmatrix} .$$

●

If $E = I$ then (4.2a) becomes, with $N = n-1$,

$$\begin{vmatrix} 0 & F^{n-1} \\ 0 & I \\ \hline 0 & HF^{n-2} \\ \vdots & \vdots \\ 0 & HF \\ 0 & H \end{vmatrix} \begin{vmatrix} w_{n-1} \\ w_0 \end{vmatrix} = \begin{vmatrix} x_n \\ Fx_0 \\ \hline \bar{y}_{1,n} \end{vmatrix} ;$$

so that $w_0 = Fx_0$. Taking into account this and (4.2b), write the above as

$$\begin{vmatrix} F^n \\ I \\ V^s_n \end{vmatrix} \triangleq \begin{vmatrix} F^n \\ I \\ \hline HF^{n-1} \\ \vdots \\ HF \\ H \end{vmatrix} x_0 = \begin{vmatrix} x_n \\ x_0 \\ \hline \bar{y}_{0,n} \end{vmatrix} , \tag{4.7}$$

. the familiar state space result. Note that the ual definitions of observability and reconstructi-ility for discrete state systems both derive from r single definition for observability of the pair $_0,x_n)$. Hence, in the usual terminology, x_0 is ervable if (4.7) has a unique solution with pect to x_0, i.e. $N(V_n^{\bar{}}) = 0$. On the other hand, is reconstructible if (4.7) has a unique solution h respect to x_n, i.e. $N(V_n^{\bar{}}) \subset N(F^n)$.

Finally, the descriptor variable boundary value onstruction problem is solved.

rollary 4.4

Let (2.1) be regular and $\bar{u}_{0,N+1} = 0$. Then the ues of Fx_0 and Ex_{N+1} can be uniquely determined m $\bar{y}_{1,N+1}$ if and only if (4.6) holds. In this e they are given by

$$\begin{vmatrix} Ex_{N+1} \\ Fx_0 \end{vmatrix} = \begin{vmatrix} F\phi_{-1} & F\phi_{N-1} \\ E\phi_{-N} & E\phi_0 \end{vmatrix} V_N^+ \bar{y}_{1,N+1} . \quad (4.8)$$

E=I this reduces to the known state-space result.

It appears in general to be impossible, or at st too complex notationally, to solve for x_0, $_1$ themselves. Work on this is in progress. It uld also be possible to show the relation between 5) and the duals of conditions (3.1) and (3.3)].

It is worthwhile to compare the notions of ervability and conditionability [1,2,16]. (2.1) conditionable if there exists a unique $\bar{y}_{0,N+1}$ for $\bar{x}_{0,N+1}$ satisfying $x_0 \in N(E)$, $x_N \in N(F)$ [16]. s occurs if and only if rank $[zE-F]$ = rank $\begin{vmatrix} zE-F \\ E \end{vmatrix}$ almost every z, and is implied by regularity. ditionability is defined in terms of uniqueness solution $\bar{y}_{0,N+1}$ of (2.7), while observability is ined in terms of (4.2).

V. ADMISSIBLE BOUNDARY CONDITIONS

Define the pair (x_0,x_N) to be an admissible ndary condition if, given (x_0,x_N), there exists a ution $\bar{x}_{0,N+1}$ to (2.1) for some $\bar{u}_{0,N}$. This is a eralization of the notion of admissible initial dition x_0 discussed in [1-6,9,10,12-14,15,17- . Such a generalization seems appropriate ause (2.1) is inherently a noncasual (i.e. non-nted) system; so that a symmetric treatment of ial and final conditions is more natural. This oach was also successful in the above treatment eachability and observability.

To characterize the set of admissible boundary litions as a subspace of R^{2n}, write the general t-squares solution to (2.7a) as

$$\bar{x}_{0,N+1} = A_N^I B_N \bar{u}_{0,N} + (I - A_N^I A_N)\bar{w}_{0,N+1} \quad (5.1)$$

arbitrary $\bar{w}_{0,N+1}$. Since (2.1) is regular, (5.1) n exact solution (i.e. $R(B_N) \subset R(A_N)$). Now we

rem 5.1

Let (2.1) be regular. Then (x_0,x_N) is an ssible boundary condition if and only if

$$\in R \begin{vmatrix} \phi_{-1}F & \phi_{N-1}F \\ \phi_{-N}E & \phi_0 E \end{vmatrix} + \begin{vmatrix} \phi_0 E & -\phi_{N-1}F \\ \phi_{-N}E & -\phi_{-1}F \end{vmatrix}^{-1} R(U_N) \quad (5.2)$$

where superscript "-1" represents inverse image.

proof:

Note that in (5.1) the first term of RHS is the input-dependent portion and the second term of RHS is the zero-input portion of the solution. Let (2.1) be regular. Then the zero-input solution supplies n additional conditions which, together with $\bar{u}_{0,N}$, specify a unique solution $\bar{x}_{0,N}$. The zero-input solution can be written as

$$\begin{vmatrix} x_N \\ \vdots \\ x_1 \\ x_0 \end{vmatrix} = (I - A_N^I A_N)\bar{w}_{0,N+1} = \begin{vmatrix} I-\phi_0 E \\ -\phi_{-1}E \\ \vdots \\ -\phi_{-N}E \end{vmatrix} \; 0 \; \begin{vmatrix} \phi_{N-1}F \\ \vdots \\ \phi_0 F \\ I+\phi_{-1}F \end{vmatrix} \bar{w}_{0,N+1} \quad (5.3)$$

since a_N^{ℓ} is a submatrix of A_N^I. Hence the zero-input solution satisfies

$$\begin{vmatrix} x_N \\ x_0 \end{vmatrix} \in R \begin{vmatrix} I- {}_0E & {}_{N-1}F \\ - {}_{-N}E & I+ {}_{-1}F \end{vmatrix} .$$

$$= R \begin{vmatrix} {}_{-1}F & {}_{N-1}F \\ {}_{-N}E & {}_0E \end{vmatrix} . \quad (5.4)$$

The input-dependent solution was found to satisfy (3.9), which represents pairs (x_0,x_N) which are "admissible for some $\bar{u}_{0,N}$".

The characterization (5.4) of the "zero-input admissible boundary conditions" is consistent with [5,10,12,13], where it is shown that ${}_0E$ is the projection onto the subspace of zero-input admissible initial conditions. It is shown in [5,12] that $- {}_{-1}F$ is the projection onto the "subspace of admissible final conditions," which is again consistent with (5.4).

If E=I then (5.2) becomes

$$\begin{vmatrix} x_N \\ x_0 \end{vmatrix} \quad R \begin{vmatrix} F^N \\ I \end{vmatrix} + [I \;\; -F^N]^{-1} R[G \;\; FG \;\; \dots \;\; F^{N-1}G] . \quad (5.5)$$

Thus for state systems with zero input any x_0 is admissible while any x_N $R(F^N)$ is admissible. The second term of this equation was discussed in connection with (3.10).

Equation (5.2) should be compared to the characterizations of the subspace of admissible x_0 by Drazin inverse in [3,9,15,22], by Weierstrass form in [4,6,17,18], by deflating subspaces in [5], by recursion relations in [12,13,23], by matrix transformations in [1,2,21], and by eigenstructure in [14].

VI. CONCLUSION

Our results show that the fundamental matrix is more important than previously realized in the study of discrete descriptor systems, and that the properties of these systems should be studied in terms of (1.1) where x_0 and x_N are the boundary values of the descriptor variable. Thus, reachable, observable, and admissible boundary value subspaces should be thought of as residing in R^{2n}, not R^n.

The descriptor open-loop control and estimation problems were solved.

Work is in progress on extending these results to continuous systems using the descriptor Cayley-Hamilton Theorem [11]

REFERENCES

[1] D. G. Luenberger, "Dynamic equations in descriptor form," IEEE Trans. Automat. Contr., pp. 312-321, June 1977.

[2] D. G. Luenberger, "Time-invariant descriptor systems," Automatica, vol. 14, pp. 473-480, 1978.

[3] S. L. Campbell, C. D. Meyer, Jr., and N. J. Rose, "Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients," SIAM J. Appl. Math., vol. 31, no. 3, pp. 411-425, Nov. 1976.

[4] E. L. Yip and R. F. Sincovec, "Solvability, Controllability, and Observability of Continuous Descriptor Systems," IEEE Trans. Automat. Contr., pp. 702-707, June 1981.

[5] F. L. Lewis, "Descriptor systems: decomposition into forward and backward subsystems," IEEE Trans. Automat. Contr., vol. AC-29, no. 1, Jan. 1984.

[6] R. F. Sincovec, A. M. Erisman, E. L. Yip, and M. A. Epton, "Analysis of descriptor systems using numerical algorithms," IEEE Trans. Automat. Contr., pp. 139-147, Feb. 1981.

[7] F. R. Gantmacher, The Theory of Matrices, New York: Chelsea, 1974.

[8] H. H. Rosenbrock, State-Space and Multivariable Theory, New York: Wiley, 1970.

[9] N. J. Rose, "The Laurent expansion of a generalized resolvent with some applications," SIAM J. Math. Analy., vol. 9, no. 4, pp. 751-758, Aug. 1978.

[10] C. E. Langenhop, "Controllability and stabilization of regular singular linear systems with constant coefficients," Dept. of Mathematics, S. Illinois Univ., Dec. 6, 1979.

[11] F. L. Lewis, "Adjoint matrix, Cayley-Hamilton Theorem, and Fadeev's method for the matrix pencil (sE-A)," Proc. 22nd IEEE Conf. Dec. and Contr., San Antonio, TX, pp. 1282-1288, 1983.

[12] C. E. Langenhop, "The Laurent expansion for a nearly singular matrix," Linear Algebra and its Applic., 4, pp. 329-340, 1971.

[13] K-T Wong, "The eigenvalue problem Tx + Sx," J. Diff. Equations, vol. 16, pp. 270-281, 1974.

[14] F. L. Lewis and K. Ozcaldiran, "The relative eigenstructure problem and descriptor systems," SIAM National Meeting, Denver, CO, June 1983.

[15] L. Pandolfi, "Controllability and stabilization for linear systems of algebraic and differential equations," JOTA, vol. 30, no. 4, pp. 601-620, April 1980.

[16] F. L. Lewis, "Descriptor systems: expanded descriptor equation and Markov parameters," IEEE Trans. Automat. Contr., vol. AC-28, no. 5, May 1983.

[17] D. Cobb, "Controllability, observability, and duality in singular systems," submitted.

[18] D. Cobb, "Feedback and pole placement in descriptor variable systems," Int. J. Control, vol. 33, no. 6, pp. 1135-1146, 1981.

[19] G. C. Verghese, B. C. Levy, and T. Kailath, "A generalized state-space for singular systems," IEEE Trans. Automat. Contr., vol. AC-26, no. 4, pp. 811-831, August 1981.

[20] G. C. Verghese and T. Kailath, "Impulsive behavior in dynamical systems: structure and significance," Proc. 4th Int. Symp. Math. Theory, Networks Syst., Delft, The Netherlands, pp. 162-168, July 1979.

[21] F. L. Lewis, "Inversion of descriptor systems," Proc. ACC, San Francisco, CA, pp. 1153-1158, June 1983.

[22] B. Dziurla and R. Newcomb, "The Drazin inverse and semi-state equations," Proc. 4th Int. Symp. Mathematical Theory of Networks and Systems, Delft, The Netherlands, pp. 283-289, July 1979.

[23] G. C. Verghese, "Further Notes on Singular Descriptions," Proc. JACC, paper TA-4, June 1981.

[24] J. H. Wilkinson, "Linear differential equations and Kronecker's Canonical Form," in Recent Advances in Numerical Analysis, C. de Boor and G. H. Golub, ed., New York: Academic, 1978.

[25] P. M. Van Dooren, "The generalized eigenstructure problem in linear system theory," IEEE Trans. Automat. Contr., vol. AC-26, no. 1, pp. 111-129, Feb. 1981.

[26] H. H. Rosenbrock, "Structural properties of linear dynamical systems," Int. J. Control, vol. 20, no. 2, pp. 191-202, 1974.

[27] P. M. Van Dooren, "The computation of Kronecker's canonical form of a singular pencil," Linear Algebra and its Applications, 27, pp. 103-140, 1979.

[28] G. Peters and J. H. Wilkinson, "Ax = Bx and the generalized eigenproblem," SIAM J. Numer. Anal., vol. 7, no. 4, pp. 479-492, Dec. 1970.

[29] P. Lancaster, "A fundamental theorem on Lambda-matrices with applications," Linear Algebra and its Applications, vol. 18, pp. 189-211, 1977.

[30] I. Gohberg and L. Rodman, "On spectral analysis of non-monic matrix and operator polynomials," Israel J. Mathematics, vol. 30, nos. 1-2, pp. 133-151, 1978.

[31] D. Cobb, "Descriptor variable systems and optimal state regulation," IEEE Trans. Automat. Contr., vol. AC-28, no. 5, pp. 601-611, May 1983.

[32] N. Karcanias and G. E. Hayton, "State-space and transfer function invariant infinite zeros: a unified approach," Proc. JACC, vol. 1, paper TA-4C, 1981.

Fundamental, Reachability, and Observability Matrices for Discrete Descriptor Systems

F. L. LEWIS

Abstract—**This paper uses the fundamental matrix of a regular discrete descriptor system to derive expressions for descriptor reachability and observability matrices. Reachable and unobservable subspaces are defined. It is shown that the natural space for analyzing descriptor system properties is R^{2n} (where n is the dimension of the system), not R^n as is the case for state-space systems. Solutions are provided for the descriptor open-loop control and estimation problems.**

I. INTRODUCTION

In this paper, we use a time-domain point of view and the descriptor fundamental matrix to define reachability and observability matrices. Our approach allows us to define reachable and unobservable subspaces in terms of the descriptor fundamental matrix. We are also able to solve the descriptor open-loop control and estimation problems.

A distinguishing feature of our work is that, in consonance with the noncausal or "nonoriented" nature of descriptor systems [22]–[24], we work with $\begin{bmatrix} x_0 \\ x_N \end{bmatrix} \in R^{2n}$ where 0, N represent initial and final times. This allows us to help clarify the duality between reachability and observability in the time domain (see also [16] which treats continuous systems). All of our results depend on the fundamental matrix; thus, no transformation to special form is required.

The focus here is on discrete systems, although the descriptor Cayley–Hamilton theorem [11] should allow a generalization to continuous systems.

At each step, it is shown how the familiar state-space results are recovered as a special case of the results presented herein.

II. FUNDAMENTAL MATRIX AND CAYLEY–HAMILTON THEOREM

Consider the linear time-invariant system over the real numbers R:

$$Ex_{k+1}=Fx_k+Gu_k \tag{2.1a}$$

$$y_k=Hx_k; \qquad k=0,\ 1,\ \cdots,\ N-1 \tag{2.1b}$$

where $u_k \in R^m$, $x_k \in R^n$, $y_k \in R^p$, and N is a finite integer defining the time interval of interest. Note that if E is nonsingular, it is possible to solve for x_k by forward iteration given x_0. On the other hand, if F is nonsingular, it is possible to solve for x_k by backward iteration given x_N. In general, however, it is not possible to solve (2.1) by simple iteration in one direction [2]–[6].

We assume throughout that (2.1) is regular, i.e., $|zE - F| \neq 0$ [1], [2], [7], [8]. In this case, we can write the unique Laurent expansion about infinity for the resolvent matrix as

$$(zE-F)^{-1}=z^{-1}\sum_{k=-\mu}^{\infty} \phi_k z^{-k} \tag{2.2}$$

where μ is the index of nilpotency of the pencil $(zE - F)$ [9]–[11]. We call $\{\phi_k\}$ the *fundamental matrix* for (2.1). It follows directly from (2.2) that ϕ_k satisfies

$$E\phi_k-F\phi_{k-1}=\delta_{ok}I \tag{2.3a}$$

$$\phi_kE-\phi_{k-1}F=\delta_{ok}I \tag{2.3b}$$

where δ_{ok} is the Kronecker delta [10], [12].

Manuscript received February 1, 1984; revised September 13, 1984. This paper is based on a prior submission of December 22, 1982. Paper recommended by Past Associate Editor, W. A. Wolovich. This work was supported by the National Science Foundation Grant under ECS-8204656.

The author is with the School of Electrical Engineering, Georgia Institute of Technology, Atlanta, GA 30332.

0018-9286/85/0500-0502$01.00

In [12] it is shown how to compute $\{\phi_k\}$ given ϕ_0 and ϕ_{-1}, and in [9] it is how to compute $\{\phi_k\}$ from E and F using the Drazin inverse. (The Drazin inverse can be computed recursively as described in [3].) It is shown [10], [12] that $\phi_0 E$ is the projection on H_I along H_N, and that $-\phi_{-1}F$ is the projection on H_N along H_I where the finite and infinite eigenspaces H_I and H_N are defined in [12]–[14]. H_I, may also be interpreted as the subspace of admissible initial conditions with zero input [1], [2], [13].

In this paper, we focus on the properties and uses of the descriptor fundamental matrix. The first result is the following. It is proved in [11].

Theorem 2.1: Let

$$\Delta(z) = |zE-F| = p_0 z^n - p_1 z^{n-1} - \cdots - p_n. \tag{2.4}$$

Then

$$p_0\phi_k - p_1\phi_{k-1} - \cdots - p_n\phi_{k-n} = 0 \tag{2.5}$$

for $k \geqslant n$ and $k \leqslant -1$. ●

If $E = I$, then $\mu = 0$ and $\phi_k = F^k$, so that (2.5) with $k = n$ becomes simply $\Delta(F) = 0$. We therefore call Theorem 2.1 the *descriptor Cayley–Hamilton theorem.*

We shall require the following notation. Write (2.1) in expanded form (c.f. [1], [2]) as

$$\begin{bmatrix} E & -F & 0 & \cdots & 0 & 0 \\ 0 & E & -F & \cdots & 0 & 0 \\ \cdot & & \ddots & \ddots & \cdot & \cdot \\ \cdot & & & \ddots & \cdot & \cdot \\ 0 & & & \ddots & -F & 0 \\ 0 & \cdots & \cdots & \cdots & E & -F \end{bmatrix} \begin{bmatrix} x_N \\ \cdot \\ \cdot \\ \cdot \\ x_1 \\ x_0 \end{bmatrix} = \begin{bmatrix} G & & 0 \\ & \ddots & \\ 0 & & G \end{bmatrix} \begin{bmatrix} u_{N-1} \\ \cdot \\ \cdot \\ u_1 \\ u_0 \end{bmatrix} \tag{2.6a}$$

$$\begin{bmatrix} y_{N-1} \\ \cdot \\ \cdot \\ y_1 \\ y_0 \end{bmatrix} = \begin{bmatrix} H & & 0 \\ & \ddots & \\ 0 & & H \end{bmatrix} \begin{bmatrix} x_{N-1} \\ \cdot \\ \cdot \\ x_1 \\ x_0 \end{bmatrix} \tag{2.6b}$$

or by appropriate definition of the coefficient matrices A_N, B_N, C_N and the input, descriptor, and output sequence vectors $\bar{u}_{0,N}$, $\bar{x}_{0,N}$, $\bar{y}_{0,N}$ as

$$A_N\bar{x}_{0,N+1} = B_N\bar{u}_{0,N} \tag{2.7a}$$

$$\bar{y}_{0,N} = C_N\bar{x}_{0,N}. \tag{2.7b}$$

Define also the auxiliary matrix

$$a_N = \begin{bmatrix} -F & 0 & \cdots & 0 & 0 \\ E & -F & & \vdots & \vdots \\ 0 & E & \ddots & \vdots & \vdots \\ \cdot & \cdot & \ddots & \ddots & \vdots \\ \cdot & \cdot & & \ddots & \vdots \\ 0 & 0 & \cdots & E & -F \\ 0 & 0 & \cdots & 0 & E \end{bmatrix}_{nN\times n(N-1)} \tag{2.8}$$

which is just A_N with the first and last block columns deleted.

The next results will subsequently be required.

Lemma 2.2: Suppose (2.1) is regular. Then a right inverse of A_N for any $N > 0$ is given by

$$A_N^r = \begin{bmatrix} \phi_0 & \phi_1 & \phi_2 & \cdots & \phi_{N-1} \\ \phi_{-1} & \phi_0 & \phi_1 & \cdots & \phi_{N-2} \\ \cdot & & \ddots & \ddots & \cdot \\ \cdot & & & \ddots & \cdot \\ \cdot & & & & \phi_0 \\ \phi_{-N} & \cdots & \cdots & \cdots & \phi_{-1} \end{bmatrix} \tag{2.9}$$

Proof: Show that $A_N A_N^r = I$ by using (2.3a). ●

Lemma 2.3: Suppose that (2.1) is regular. Then a left inverse of a_N for any $N > 0$ is given by

$$a_N^l = \begin{bmatrix} \phi_{-1} & \phi_0 & \phi_1 & \cdots & \phi_{N-2} \\ \phi_{-2} & \phi_{-1} & \phi_0 & \cdots & \vdots \\ \cdot & & \ddots & \ddots & \vdots \\ \cdot & & & \ddots & \vdots \\ \phi_{-N+1} & \cdots & \cdots & \phi_{-1} & \phi_0 \end{bmatrix} \tag{2.10}$$

Proof: Show that $a_N^l a_N = I$ by using (2.3b). ●

It is well known [1], [2] that regularity of (2.1) is equivalent to the full ranks of A_N and a_N, and hence of A_N^r and a_N^l, for all $N > 1$. See [20] where constructions similar to (2.9), (2.10) are given in terms of a Drazin inverse.

III. Reachability

Given $z_1, z_2, \in R^n$, define the pair (z_1, z_2) to be *reachable* if there exists a control $\bar{u}_{0,N}$ for some $N > 0$ such that $\bar{x}_{0,N+1}$ is a solution to (2.1a) with $x_0 = z_1$, $x_N = z_2$. We will loosely speak of (x_0, x_N) as reachable. Define system (2.1) to be reachable if all pairs (x_0, x_N) are reachable. This is consistent with definitions in [4], [10], [15], and [17].

It is well known [4] that (2.1) is reachable if and only if rank $[zE - F\ G] = n$ for all z and rank $[E\ G] = n$.

The next result provides the basis for our discussion of reachability in terms of the fundamental matrix.

Theorem 3.1: Let (2.1) be regular and define the *descriptor reachability matrix*

$$U_N = \begin{bmatrix} \phi_0 G & \phi_1 G & \cdots & \phi_{N-1}G \\ \phi_{-N}G & \cdots & \phi_{-2}G & \phi_{-1}G \end{bmatrix}. \tag{3.1}$$

Then over any interval $[0, N]$, the control sequence and the initial and final values of the descriptor variable are related by

$$\begin{bmatrix} \phi_0 E & -\phi_{N-1}F \\ \phi_{-N}E & -\phi_{-1}F \end{bmatrix} \begin{bmatrix} x_N \\ x_0 \end{bmatrix} = U_N\bar{u}_{0,N}. \tag{3.2}$$

Furthermore, reachability for (2.1) can be studied in terms of (3.2).

Proof: Rewrite (2.6a), (2.7a) as

$$\begin{bmatrix} -Ex_N \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ Fx_0 \end{bmatrix} = [a_N\ B_N] \begin{bmatrix} \bar{x}_{1,N} \\ -\bar{u}_{0,N} \end{bmatrix}. \tag{3.3}$$

Premultiply both sides of (3.3) by A_N^r as given by (2.9) and use (2.3b) to obtain

$$\begin{bmatrix} -\phi_0 E & \phi_{N-1}F \\ \hline -\phi_{-1}E & \phi_{N-2}F \\ \cdot & \\ \cdot & \\ \cdot & \\ -\phi_{-N+1}E & \phi_0 F \\ \hline -\phi_{-N}E & \phi_{-1}F \end{bmatrix} \begin{bmatrix} x_N \\ x_0 \end{bmatrix}$$

$$= \left[\begin{array}{c|cccc} 0 & \phi_0 G & \phi_1 G & \cdots & \phi_{N-1}G \\ \hline & \phi_{-1}G & \phi_0 G & \cdots & \phi_{N-2}G \\ & \cdot & \ddots & & \cdot \\ I & \vdots & & \ddots & \vdots \\ & \cdot & & & \cdot \\ & \phi_{-N+1}G & \cdots & \cdots & \phi_0 G \\ \hline 0 & \phi_{-N}G & \cdots & \phi_{-2}G & \phi_{-1}G \end{array}\right] \begin{bmatrix} \bar{x}_{1,N} \\ -\bar{u}_{0,N} \end{bmatrix} \tag{3.4}$$

Since A_N^r has full column rank, (3.3) has a solution if and only if (3.4) does. The form of (3.4) guarantees that we can solve for $\bar{x}_{1,N}$ given any x_0, x_N, and $\bar{u}_{0,N}$ which are otherwise consistent with (3.4). Therefore, (3.3) is effectively equivalent to (3.2). ●

In (3.1), $\phi_k = 0$ for $k < -\mu$. For ease of notation, this is not explicitly indicated.

From this theorem, there follow several results whose proofs are quite trivial. $R(\cdot)$ denotes the range of a linear operator, and the superscripts "-1" and "$+$" denote the inverse image of a linear operator and Moore–Penrose matrix inverse, respectively. The first result provides another reachability test.

Corollary 3.2: Let (2.1) be regular. Then the pair (x_0, x_N) is reachable if and only if

$$\begin{bmatrix} x_N \\ x_0 \end{bmatrix} \in \begin{bmatrix} \phi_0 E & -\phi_{N-1}F \\ \phi_{-N}E & -\phi_{-1}F \end{bmatrix}^{-1} R(U_N) \tag{3.5}$$

and the system is reachable if and only if

$$R\left(\begin{bmatrix} \phi_0 E & \phi_{N-1}F \\ \phi_{-N}E & \phi_{-1}F \end{bmatrix}\right) \subset R(U_N). \tag{3.6}$$ ●

Equation (3.5) characterizes the reachable subspace, and it should be compared to the equivalent characterization by deflating subspaces in [19] and by the Weierstrass form in [4].

If $E = 1$, then with $N = n$, (3.2) becomes

$$\begin{bmatrix} I & -F^n \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_0 \end{bmatrix} = \begin{bmatrix} G & FG & \cdots & F^{n-1}G \\ 0 & 0 & \cdots & 0 \end{bmatrix} \bar{u}_{0,n} \tag{3.7}$$

which is equivalent to $(x_n - F^n x_0) \in (R[G\ FG \cdots F^{n-1}G]$, the familiar state system result. Note that the usual definitions of reachability and controllability for discrete state systems both derive from our single definition for reachability of the pair (x_0, x_n). Hence, in the usual terminology, a state system is reachable if $x_n \in R[G\ FG \cdots F^{n-1}G]$ for all $x_n \in R^n$, and is controllable if $F^n x_0 \in R[G\ FG \cdots F^{n-1}G]$ for all $x_0 \in R^n$.

The open-loop control problem for descriptor systems is solved next.

Corollary 3.3: Let N be fixed. A control on $[0, N]$ which makes the descriptor variable of the regular system (2.1) take on prescribed values x_0 and x_N exists if and only if (3.5) holds, and then it is given by

$$\bar{u}_{0,N} = U_N^+ \begin{bmatrix} \phi_0 E & -\phi_{N-1}F \\ \phi_{-N}E & -\phi_{-1}F \end{bmatrix} \begin{bmatrix} x_N \\ x_0 \end{bmatrix}. \tag{3.8}$$ ●

This reduces to the well-known state-space result in the case $E = I$.

IV. Observability

Observability for descriptor systems seems to be a more difficult problem conceptually than is reachability. We deal with a form of observability which gives dual results in the time domain to those of Section III, and which reduces if $E = I$ to the state-space results. Our definition is similar to that in [16] and [18] and should be contrasted to the definition in [4].

Given a fixed $N > 1$ and $z_1, z_2 \in R^n$, (z_1, z_2) is an *admissible* pair of boundary values on $[0, N+1]$ if, when $\bar{u}_{0,N+1} = 0$, there is a solution to (2.1) with $x_0 = z_1$, $x_{N+1} = z_2$. Given a fixed $N > 1$, an admissible pair (z_1, z_2) is said to be *observable* if, when $\bar{u}_{0,N+1} = 0$, knowledge of the output $\bar{y}_{1,N+1}$ resulting when $x_0 = z_1$ and $x_{N+1} = z_2$ is sufficient to uniquely determine Fx_0, Ex_{N+1}. We shall loosely speak of (x_0, x_{N+1}) as observable. Define system (2.1) to be observable if all admissible pairs (x_0, x_{N+1}) are observable. It will become clear that the choice of final time (i.e., $N+1$ not N) will result in a duality with the results of Section III.

The next result provides the basis for our discussion of observability. It is the dual to Theorem 3.1, and it is proven in a similar manner [21], except that it uses Lemma 2.3 instead of Lemma 2.2.

Theorem 4.1: Let (2.1) be regular and define the *descriptor observability matrix*

$$V_N = \begin{bmatrix} H\phi_{-1} & H\phi_{N-1} \\ H\phi_{-2} & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & H\phi_1 \\ H\phi_{-N} & H\phi_0 \end{bmatrix} \tag{4.1}$$

Then over any fixed interval $[0, N+1]$, the output and the initial and final values of the descriptor variable are related by

$$\begin{bmatrix} F\phi_{-1} & F\phi_{N-1} \\ E\phi_{-N} & E\phi_0 \\ \hline \multicolumn{2}{c}{V_N} \end{bmatrix} \begin{bmatrix} w_N \\ w_0 \end{bmatrix} = \begin{bmatrix} Ex_{N+1} \\ Fx_0 \\ \hline \bar{y}_{1,N+1} \end{bmatrix} \tag{4.2a}$$

$$\begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} x_{N+1} \\ x_0 \end{bmatrix} = \begin{bmatrix} y_{N+1} \\ y_0 \end{bmatrix} \tag{4.2b}$$

where w_0, w_N are intermediate variables. Furthermore, the observability of (2.1) can be studied in terms of (4.2). ●

At this point, the next results are immediate.

Corollary 4.2: A regular system (2.1) is observable if and only if, for some $N > 0$,

$$\mathfrak{N}(V_N) \subset \mathfrak{N}\begin{bmatrix} F\phi_{-1} & F\phi_{N-1} \\ E\phi_{-N} & E\phi_0 \end{bmatrix} \tag{4.3}$$

where $\mathfrak{N}(\cdot)$ represents the null space of a linear operator.

Proof: For a given $\bar{y}_{1,N+1}$, the solution to (4.2a) is unique with respect to Fx_0, Ex_{N+1} if and only if (4.3) holds. ●

The observable subspace is characterized next.

Corollary 4.3: Let (2.1) be regular. Then an admissible pair (x_0, x_{N+1}) makes a zero contribution to the output sequence $\bar{y}_{1,N+1}$ if and only if

$$\begin{bmatrix} Ex_{N+1} \\ Fx_0 \end{bmatrix} \in \begin{bmatrix} F\phi_{-1} & F\phi_{N-1} \\ E\phi_{-N} & E\phi_0 \end{bmatrix} \mathfrak{N}(V_N). \tag{4.4}$$

Proof: Any $\left[\begin{smallmatrix} w_N \\ w_0 \end{smallmatrix}\right] \in \mathfrak{N}(V_N)$ results in $\bar{y}_{1,N+1} = 0$. Such "unobservable intermediate variables" contribute a component to the solution of (4.2) given by

$$\begin{bmatrix} Ex_{N+1} \\ Fx_0 \end{bmatrix} = \begin{bmatrix} F\phi_{-1} & F\phi_{N-1} \\ E\phi_{-N} & E\phi_0 \end{bmatrix} \begin{bmatrix} w_N \\ w_0 \end{bmatrix}$$ ●

If $E = I$, then (4.2a) becomes, with $N = n - 1$,

$$\begin{bmatrix} 0 & F^{n-1} \\ 0 & I \\ \hline 0 & HF^{n-2} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & HF \\ 0 & H \end{bmatrix} \begin{bmatrix} \mathscr{N}_{n-1} \\ w_0 \end{bmatrix} = \begin{bmatrix} x_n \\ Fx_0 \\ \hline \bar{y}_{1,n} \end{bmatrix}$$

so that $w_0 = Fx_0$. Taking into account this and (4.2b), write the above as

$$\begin{bmatrix} F^n \\ I \\ \hline V_n^s \end{bmatrix} x_0 \triangleq \begin{bmatrix} F^n \\ I \\ \hline HF^{n-1} \\ \cdot \\ \cdot \\ \cdot \\ HF \\ H \end{bmatrix} x_0 = \begin{bmatrix} x_n \\ x_0 \\ \hline \bar{y}_{0,n} \end{bmatrix}, \tag{4.5}$$

i.e., the familiar state-space result. Note that the usual definitions of observability and reconstructibility for discrete state systems both derive from our single definition for observability of the pair (x_0, x_n). Hence, in the usual terminology, x_0 is observable if (4.5) has a unique solution with respect to x_0, i.e., $\mathfrak{N}(V_n^s) = 0$. On the other hand, x_n is reconstructible if (4.5) has a unique solution with respect to x_n, i.e., $\mathfrak{N}(V_n^s) \subset \mathfrak{N}(F^n)$.

Finally, the descriptor variable boundary value reconstruction problem is solved.

Corollary 4.4: Let (2.1) be regular and $\bar{u}_{0,N+1} = 0$. Then the values of Fx_0 and Ex_{N+1} can be uniquely determined from $\bar{y}_{1,N+1}$ for all admissible (x_0, x_{N+1}) if and only if (4.3) holds. In this case, they are given by

$$\begin{bmatrix} Ex_{N+1} \\ Fx_0 \end{bmatrix} = \begin{bmatrix} F\phi_{-1} & F\phi_{N-1} \\ E\phi_{-N} & E\phi_0 \end{bmatrix} V_N^+ \bar{y}_{1,N+1}. \tag{4.6}$$ ●

If $E = I$, this reduces to the known state-space result.

V. Conclusion

We have presented tests for reachability and observability in descriptor systems which depend on matrices constructed from the fundamental matrix. These matrices reduce to the reachability and observability matrices in the state-space case. The descriptor open-loop control and boundary-value reconstruction problems were solved. It was shown that the reachable and unobservable subspaces should be thought of as residing in R^{2n}, not R^n, where n is the dimension of the descriptor variable.

Acknowledgment

The author wishes to thank the reviewers who offered many suggestions for improving this paper.

References

[1] D. G. Luenberger, "Dynamic equations in descriptor form," *IEEE Trans. Automat. Contr.*, pp. 312-321, June 1977.

[2] ——, "Time-invariant descriptor systems," *Automatica*, vol. 14, pp. 473-480, 1978.

[3] S. L. Campbell, C. D. Meyer, Jr., and N. J. Rose, "Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients," *SIAM J. Appl. Math.*, vol. 31, pp. 411-425, Nov. 1976.

[4] E. L. Yip and R. F. Sincovec, "Solvability, controllability, and observability of continuous descriptor systems," *IEEE Trans. Automat. Contr.*, pp. 702-707, June 1981.

[5] F. L. Lewis, "Descriptor systems: Decomposition into forward and backward subsystems," *IEEE Trans. Automat. Contr.*, vol. AC-29, Jan. 1984.

[6] R. F. Sincovec, A. M. Erisman, E. L. Yip, and M. A. Epton, "Analysis of descriptor systems using numerical algorithms," *IEEE Trans. Automat. Contr.*, pp. 139-147, Feb. 1981.

[7] F. R. Gantmacher, *The Theory of Matrices*. New York: Chelsea, 1974.

[8] H. H. Rosenbrock, *State-Space and Multivariable Theory*. New York: Wiley, 1970.

[9] N. J. Rose, "The Laurent expansion of a generalized resolvent with some applications," *SIAM J. Appl. Math. Anal.*, vol. 9, pp. 751-758, Aug. 1978.

[10] C. E. Langenhop, "Controllability and stabilization of regular singular linear systems with constant coefficients," Dept. Math., Southern Illinois Univ., Carbondale, Dec. 6, 1979.

[11] F. L. Lewis, "Ajoint matrix, Cayley-Hamilton theorem, and Fadeev's method for the matrix pencil (sE-A)," in *Proc. 22nd IEEE Conf. Decision Contr.*, San Antonio, TX, 1983, pp. 1282-1288.

[12] C. E. Langenhop, "The Laurent expansion for a nearly singular matrix," *Linear Algebra Appl.*, vol. 4, pp. 329-340, 1971.

[13] K.-T. Wong, "The eigenvalue problem $\lambda Tx + Sx$," *J. Diff. Eq.*, vol. 16, pp. 270-281, 1974.

[14] F. L. Lewis and K. Ozcaldiran, "The relative eigenstructure problem and descriptor systems," presented at the SIAM Nat. Meet., Denver, CO, June 1983.

[15] L. Pandolfi, "Controllability and stablization for linear systems of algebraic and differential equations," *J. Optimiz. Theory Appl.*, vol. 30, pp. 601-620, Apr. 1980.

[16] D. Cobb, "Controllability, observability, and duality in singular systems," submitted for publication.

[17] E. N. Khasina, "Control of singular linear dynamic systems," *Automat. Remote Contr.*, vol. 43, pp. 448-455, Apr. 1982.

[18] G. C. Verghese and T. Kailhath, "Impulsive behavior in dynamical systems: Structure and significance," in *Proc. 4th Int. Symp. Math. Theory, Networks Syst.*, Delft, The Netherlands, July 1979, pp. 162-168.

[19] P. M. Van Dooren, "The generalized eigenstructure problem in linear system theory," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 111-129, Feb. 1981.

[20] S. L. Cambell, *Singular Systems of Differential Equations*. San Francisco, CA: Pitman, 1980.

[21] F. L. Lewis, "Descriptor systems: Fundamental matrix, reachability and observability matrices, subspaces," in *Proc. 23rd IEEE Conf. Decision Contr.*, Dec. 1984.

[22] D. G. Luenberger, "Nonlinear descriptor systems," *J. Econ. Dyn. Contr.*, vol. 1, pp. 219-242, 1979.

[23] S. L. Campbell and C. D. Meyer, Jr., *Generalized Inverses of Linear Transformations*. London, England: Pitman, 1979.

[24] J. D. Aplevich, "Time domain input-output representations of linear systems," *Automatica*, vol. 17, no. 3, pp. 509-522, 1981.

Proceedings of 24th Conference
on Decision and Control
Ft. Lauderdale, FL • December 1985

ON THE EIGENSTRUCTURE ASSIGNMENT OF SINGULAR SYSTEMS

F. L. Lewis and K. Ozcaldiran
School of Electrical Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332-0250
(404) 894-2994

Research Supported by NSF Grant ECS-8204656

ABSTRACT

We examine the feedback assignment of eigenstructure for linear time-invariant singular systems. The possible closed-loop structure is investigated using the reachable and supremal $(A,E,\mathbf{R}(B))$-invariant subspaces. No controllability assumptions are made.

It is shown how to determine in terms of the original system matrices exactly what is possible using feedback; no transformation to special form is needed. A method is given to compute the required feedback gain.

1. BACKGROUND

The problem of feedback assignment of eigenstructure in state systems has received much attention since the seminal paper by Moore [1]. See [2-5]. In this paper, we examine eigenstructure assignment for singular systems which have the form

$$E\dot{x} = Ax + Bu , \tag{1.1}$$

$x \in R^n$, $u \in R^m$. We assume the regular case, $|sE-A| \neq 0$.

An extension of Moore's method to (1.1) has been presented in [13]; here we consider some further results which clarify the structure of the problem. We also consider the case of an unreachable system. Any proofs which are omitted may be found in [11].

We call (1.1) <u>reachable</u> if for all $z \in R^n$ there is a control $u(t)$ such that the solution $x(t)$ is continuously differentiable and satisfies $x(0) = 0, x(T) = z$ for some $T > 0$ [6]. The <u>reachable subspace</u> $\mathbf{R}$ is the subspace of all $x(T)$ reachable from $x(0) = 0$. We call (1.1) <u>controllable</u> if for all $z \in R^n$ there is a control $u(t)$ such that the solution $x(t)$ is continuously differentiable and satisfies $x(0) = z$, $x(T) = 0$ for some $T > 0$. This is equivalent [11] to modal controllability in [7]. The triple (E,A,B) is reachable if and only if the <u>reachability pencil</u>

$$P(s) = [sE-A \quad B] \tag{1.2}$$

has no finite zeros and

$$\text{rank } [E \quad B] = n . \tag{1.3}$$

The triple is controllable if and only if $P(s)$ has no finite or infinite zeros. ($P(s)$ has no infinite zeros if $P(1/s)$ has no zeros at $s = 0$.)

Although all our results are in terms of $E,A,$ and B, proofs are more conveniently couched in terms of the Weierstrass form of (1.1),

$$\dot{x}_1 = Jx_1 + B_1u \tag{1.4a}$$

$$N\dot{x}_2 = x_2 + B_2u \tag{1.4b}$$

with $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$ where $n_1 = \deg|sE-A|$, and J, N in Jordan form with N nilpotent with index α. In terms of (1.4) we have [6]

$$\mathbf{R} = \langle J|B_1\rangle \oplus \langle N|B_2\rangle \stackrel{\Delta}{=} \mathbf{R}_f \oplus \mathbf{R}_\infty , \tag{1.5}$$

where

$$\langle J|B_1\rangle \stackrel{\Delta}{=} \mathbf{R}[B_1 \quad JB_1 \quad \dots \quad J^{n_1-1}B_1] . \tag{1.6}$$

and $\mathbf{R}(\cdot)$ denotes range.

To compute $\mathbf{R}$, we may use the subspace recursions

$$\mathbf{X}_{k+1}(\mathbf{K}) = \mathbf{K} \cap A^{-1}\left(E\mathbf{X}_k + \mathbf{R}(B)\right) \tag{1.7}$$

$$\mathbf{Y}_{k+1}(\mathbf{K}) = \mathbf{K} \cap E^{-1}\left(A\mathbf{Y}_k + \mathbf{R}(B)\right) \tag{1.8}$$

where $\mathbf{K}$ is a given subspace and superscript "-1" denotes inverse image. For if $\mathbf{X}_0 = R^n$ and $\mathbf{Y}_0 = 0$, then [11]

$$\mathbf{R} = \mathbf{X}_n(R^n) \cap \mathbf{Y}_n(R^n) \tag{1.9a}$$

$$= \mathbf{X}_n\left(\mathbf{Y}_n(R^n)\right) . \tag{1.9b}$$

Also of interest to us is the <u>supremal $(A,E,\mathbf{R}(B))$-invariant subspace</u> [12]

$$\mathbf{V}^* = \sup\left\{\mathbf{S} \subset R^n \middle| A\mathbf{S} \subset E\mathbf{S} + \mathbf{R}(B)\right\} \tag{1.10}$$

which may be computed [11] using

$$\mathbf{V}^* = \mathbf{X}_n(R^n) \tag{1.11}$$

with $\mathbf{X}_0 = R^n$ in (1.7). In terms of the Weierstrass form, $\mathbf{V}^*$ is given by

$$\mathbf{V}^* = R^{n_1} \oplus \mathbf{R}_\infty \tag{1.12}$$

(which is an intermediate step in the proof of (1.9)). Note that

$$\mathbf{R} \subset \mathbf{V}^* . \tag{1.13}$$

It can be shown [11] that $\mathbf{R}$ and $\mathbf{V}^*$ are invariant under the feedback

$$u = Fx + r , \tag{1.14}$$

with $r(t)$ an auxiliary input.

2. FINITE EIGENSTRUCTURE

Determine a minimal basis for the null space of $P(s)$, so that

$$[sE-A \quad B]\begin{bmatrix} L(s) \\ M(s) \end{bmatrix} = 0 . \tag{2.1}$$

Defining a feedback gain by

$$v_i \stackrel{\Delta}{=} L(\lambda_i)w_i , \quad u_i \stackrel{\Delta}{=} L(\lambda_i)w_i \tag{2.2}$$

$$Fv_i = -u_i \tag{2.3}$$

CH2245-9/85/0000-0179 $1.00 © 1985 IEEE

for some $\lambda_i \epsilon C$, the complex numbers, and $w_i \epsilon C^m$ there results

$$(\lambda_i E - (A+BF))v_i = 0 \tag{2.4}$$

so that λ_i is an eigenvalue of the closed-loop system with eigenvector v_i.

The closed-loop initial manifold [9] is $H_{IF} = \mathcal{X}_n$ in

$$\mathcal{X}_{k+1} = (A+BF)^{-1} E \mathcal{X}_k \tag{2.5}$$

with $\mathcal{X}_0 = R^n$. It is easy to show

$$v_i \epsilon H_{IF} , \tag{2.6}$$

for clearly $v_i \epsilon \mathcal{X}_0$. If $v_i \epsilon \mathcal{X}_k$, then by (2.4)

$$(A+BF)v_i = \lambda_i E v_i \subset E\mathcal{X}_k \tag{2.7}$$

whence

$$v_i \epsilon \mathcal{X}_{k+1} .$$

If v_i is complex, then we also select $w_j = w_i^*$, $\lambda_j = \lambda_i^*$ in (2.2), (2.3) so that $v_j = v_i^*$ is also a closed-loop eigenvector and F is real [1]. In that event, we interpret $\text{span}\{v_i, v_i^*\}$ as $\text{span}\{\text{Re}(v_i), \text{Im}(v_i)\}$ so that we may talk about subspaces of R^n.

To find in terms independent of F a subspace to which v_i belongs, note that

$$H_{IF} \subset V^* \qquad \text{for all } F ; \tag{2.8}$$

for by (2.5)

$$(A+BF)H_{IF} \subset EH_{IF} , \tag{2.9}$$

so that if $y \epsilon H_{IF}$, then $(A+BF)y = Ez$ for some $z \epsilon H_{IF}$. Hence, $Ay = Ez - BFy$ or

$$AH_{IF} \subset EH_{IF} + R(B) , \tag{2.10}$$

whence follows (2.8) (c.f. [14]). Therefore, for any choice of λ_i, w_i evidently

$$v_i \epsilon V^* , \tag{2.11}$$

which provides a characterization of the possible closed-loop eigenvectors. A more detailed characterization is not difficult to deduce by the following line of thought.

In Weierstrass form, (2.1) becomes

$$\begin{bmatrix} sI-J & 0 \\ 0 & sN-I \end{bmatrix} L(s) = - \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} M(s) . \tag{2.12}$$

For values of $\lambda \epsilon C$ such that $|\lambda E - A| \neq 0$, we may select $M(\lambda) = -I_m$ and so

$$L(\lambda) = \begin{bmatrix} (\lambda I - J)^{-1} B_1 \\ (\lambda N - I)^{-1} B_2 \end{bmatrix} \tag{2.13}$$

$$= \sum_{i=0}^{n_1 - 1} \beta_i(\lambda) \begin{bmatrix} J^i B_1 \\ 0 \end{bmatrix} + \sum_{i=0}^{n_2 - 1} \lambda^i \begin{bmatrix} 0 \\ N^i B_2 \end{bmatrix} , \tag{2.14}$$

where coefficients $\beta_i(s)$ are related to the coefficients of

$$\Delta(s) = |sE - A| \tag{2.15}$$

See [11,15]. Although

$$[L(\lambda)] \overset{\Delta}{=} \{v \epsilon C^n \mid v = L(\lambda)w \quad \text{for some } \lambda \epsilon C \text{ and } w \epsilon C^m\} \tag{2.16}$$

is not a subspace, it is nearly one. Clearly,

1. $[L(\lambda)] \subset R$, (2.17)

and we can also show that

2. the number of linearly independent vectors v in $[L(\lambda)]$ is equal to $\dim(R)$. (2.18)

The closed-loop system

$$E\dot{x} = (A+BF)x + Br \tag{2.19}$$

is regular if and only if [9]

$$N(E) \cap H_{IF} = 0 , \tag{2.20}$$

which demands

$$\{Ev_i\}_1^\rho \text{ linearly independent} , \tag{2.21}$$

with $\{v_i\}_1^\rho$ the set of assigned closed-loop eigenvectors. We have just shown how to assign $\dim\{ER\}$ eigenvectors

$$v_i \epsilon R \tag{2.22}$$

satisfying this condition. If P(s) has no finite zeros, then in terms of the Weierstrass form, (1.4a) is reachable and $R_f = R^{n_1}$, so that

$$R = V^* . \tag{2.23}$$

Otherwise, we may still select a maximum of

$$\rho_M \overset{\Delta}{=} \dim(EV^*) \tag{2.24}$$

finite closed-loop eigenvectors v_i satisfying (2.21) by considering the finite unreachable subspace, which has dimension

$$\dim(V^*) - \dim(R) , \tag{2.25}$$

as follows.

For simplicity, assume μ is a simple finite unreachable eigenvalue of (1.1). Then (c.f. [1]) for some nonzero $p_\mu \epsilon C^n$

$$p_\mu^T [\mu E - A \quad B] = 0 \tag{2.26}$$

whence for all F

$$p_\mu^T (A+BF) = p_\mu^T A = \mu p_\mu^T E , \tag{2.27}$$

so that p_μ is a left eigenvector of (E,A) which is invariant under feedback. For any $\lambda \epsilon C$ and $v \epsilon R(L(\lambda))$ there is an F (see (2.2), (2.3)) such that $(A+BF)v = \lambda Ev$ so that

$$\mu p_\mu^T Ev = p_\mu^T (A+BF)v = \lambda p_\mu^T Ev , \tag{2.28}$$

and for $\lambda \neq \mu$ it follows that $p_\mu^T EL(\lambda) = 0$.

Thus, by selecting v_μ according to (2.2) such that

$$p_\mu^T E v_\mu \neq 0 \tag{2.29}$$

we guarantee that Ev_μ is linearly independent of $\{Ev_i\}$ where the v_i correspond to selected λ_i which are not in the (open-loop) spectrum of (E,A). To see that such a v_μ exists, note that $p_\mu \epsilon R^\perp$, so that (2.29) requires Ev_μ to have a component outside R. Now let $p_\mu^T = [p_f^T \quad p_\infty^T]$ and examine the Weierstrass form of (2.26) (c.f. (2.12)) to see that $p_\infty = 0$. Hence $p_\mu \epsilon R^{n_1}$. However, E is nonsingular on R^{n_1}. In fact, the number of linearly independent vectors of the form Ev_μ with components in R^{n_1} yet outside R_f is exactly (2.25), so that by considering the reachable modes (whose assigned eigenvectors satisfy (2.22)) and the unreachable modes (whose assigned eigenvectors satisfy (2.29)), we may assign in all $\dim(EV^*)$ finite closed-loop eigenvectors v_i which satisfy (2.4) and (2.21). All of these v_i also satisfy (2.11).

3. INFINITE EIGENSTRUCTURE

Although we can now choose a feedback F which yields $\rho_M = \dim(EV^*)$ finite eigenvalues λ_i with associated characteristic vectors v_i, our feedback will assign every other number to the closed-loop spectrum as well unless the regularity condition (2.20) holds. This is equivalent to the requirement that

$$\Delta^{c\ell}(s) \triangleq |sI - (A+BF)| \tag{3.1}$$

be nonzero [8,10,13]. To guarantee this, the definition (2.3) of F must be suitably extended to R^n. This is easy to do if (1.1) is controllable at infinity. For generality, we do not assume this.

The solution to (1.4b) is [10]

$$x_2(t) = -\sum_{i=1}^{\alpha-1} \delta^{(i-1)}(t)N^i x_2(0) - \sum_{i=0}^{\alpha-1} N^i B_2 u^{(i)}(t) \tag{3.2}$$

with $\delta(t)$ the unit impulse and superscript (i) the i-th distributional derivative. Therefore, if (1.1) has an eigenvector chain of length $\alpha_i > 1$ at infinity so that

$$\begin{aligned} &Ev_i^1 = 0 \\ &Ev_i^{k+1} = Av_i^k \quad ; \quad k = 1,2,\ldots,\alpha_i-1 \\ &Av_i^{\alpha_i} \notin R(E) \end{aligned} \tag{3.3}$$

then, depending on $x_2(0)$, it can display impulsive behavior at t = 0 in the directions v_i^k; k = 1,2,..., α_i-1. To minimize the degree of the impulsive behavior, we should minimize the lengths of the chains at infinity in the closed-loop system. The following method of accomplishing this suggests itself.

In recursion (1.8), set $K = R^n$, $Y_0 = N(E)$. Find the first value K of k such that

$$AY_K + R(B) + R(E) = R^n . \tag{3.4}$$

Define

$$n \triangleq \dim\big(N(E)\big) . \tag{3.5}$$

Then, according to (1.8) and (3.4) there exist linearly independent vectors $v_i^k \epsilon R^n$ and vectors $w_i^k \epsilon R^m$, $\xi_i \epsilon R^n$ such that

<u>for i = 1,...,n:</u>

$$Ev_i^1 = 0 \tag{3.6a}$$

$$Ev_i^{k+1} = Av_i^k + Bw_i^k \quad ; \quad k = 1,\ldots,d_i-1 \tag{3.6b}$$

$$\xi_i = Av_i^{d_i} + Bw_i^{d_i} \tag{3.6c}$$

$$\operatorname{span}\{\xi_i\}_1^n + R(E) = R^n \tag{3.6d}$$

and d_i are chosen as small as possible with

$$d_i \le K+1 \quad ; \quad i = 1,\ldots,n . \tag{3.7}$$

Defining feedback F on $\operatorname{span}\{v_i^k\}$ by

$$Fv_i^k = w_i^k \quad ; \quad k = 1,\ldots,d_i \quad ; \quad i = 1,\ldots,n \tag{3.8}$$

results in

$$\begin{aligned} &Ev_i^1 = 0 \\ &Ev_i^{k+1} = (A+BF)v_i^k \quad , \quad k = 1,\ldots,d_i-1 \\ &(A+BF)v_i^{d_i} = \xi_i \end{aligned} \tag{3.9}$$

so that the v_i^k are closed-loop eigenvectors at ∞. By construction, the closed-loop chains have minimum lengths d_i, all less than or equal to K+1.

The closed-loop <u>final manifold</u> [9] is $H_{NF} = Y_n$ in

$$Y_{k+1} = E^{-1}(A+BF)Y_k \tag{3.10}$$

with $Y_0 = 0$, and it is trivial to show that

$$v_i^k \epsilon Y_k \quad ; \quad i = 1,\ldots,n . \tag{3.11}$$

4. COMBINED EIGENSTRUCTURE

The next theorem pulls together our results, showing how to choose a feedback gain to assign a desired closed-loop eigenstructure which also guarantees regularity and minimizes the degree of the remaining impulsive behavior.

<u>**Theorem 4.1**</u>

Let $\{\lambda_i\}_1^{\rho_M}$ be a self-conjugate set of complex numbers, and $\{v_i\}_1^{\rho_M}$ be selected as in (2.1)-(2.2), with $w_j = w_i^*$ when $\lambda_j = \lambda_i^*$. Choose $\{\{v_i^k\}_{k=1}^{d_i}\}_{i=1}^n$ as in (3.4)-(3.6), with $d_j = d_i$, $v_j^k = (v_i^k)^*$ if v_i^k is complex. Suppose $\{Ev_i\}_1^{\rho_M} \cup \{\{Ev_i^k\}_{k=2}^{d_i}\}_{i=1}^n$ are linearly independent, and $\{v_i\}_1^{\rho_M} \cup \{\{v_i^k\}_{k=1}^{d_i}\}_{i=1}^n$ are linearly independent. Then, with F defined by (2.3), (3.8):

a. The λ_i are the finite eigenvalues of (E,A-BF) with eigenvectors v_i,
b. The v_i^k are the infinite eigenvectors of (E,A-BF), forming n chains at ∞ of lengths d_i,
c. The closed-loop system (E,A-BF) is regular,
d. The closed-loop system has $\sum_{i=1}^{n} (d_i-1)$ impulsive directions.

<u>**Proof:**</u>

The only part left to prove is c. Form right and left modal matrices

$$V = \left[v_i \mid v_i^k\right] \tag{4.1a}$$

$$W = \left[Ev_i \mid (A+BF)v_i^k\right] \tag{4.1b}$$

with (i,k) increasing in odometer order to a maximum of (n,d_i). Then, by hypothesis V has full rank. By hypothesis, and using also (3.6d) and (3.9), W has full rank. But now $W^{-1}\big(\lambda E - (A+BF)\big)V$ takes on a Weierstrass form, which implies regularity. ●

Controllability at infinity significantly simplifies things. First note the next result, which does not seem to be well known in this general form (see [13], and [11] where it is expressed in Weierstrass form).

Lemma 4.2:

(E,A,B) is controllable at infinity if and only if

$$\mathbf{AN}(E) + \mathbf{R}(E) + \mathbf{R}(B) = R^n . \tag{4.2}$$

Proof:

The zeros of $P(s)$ at infinity are those of $P(1/s)$ at $s = 0$. Apply a nonsingular transformation to $P(s)$ to obtain

$$P(s)\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} = \left[s\bar{E}-\bar{A} \quad -\tilde{A} \;\middle|\; B \right] \tag{1}$$

with $\bar{E}$ of full column rank r. The zeros of $P(\frac{1}{s})$ are those of $[\frac{1}{s}\bar{E}-\bar{A} \quad -\tilde{A} \quad B]$ for which a minimal realization (c.f. [16]) is

$$\left[\begin{array}{cc|ccc} \multicolumn{2}{c|}{sI} & I & 0 & 0 \\ \hline -\bar{E} & s\bar{A} & 0 & \tilde{A} & B \end{array}\right] . \tag{2}$$

This has no zeros at $s = 0$ if and only if

$$\left[\bar{E} \quad \tilde{A} \quad B\right] \tag{3}$$

has full rank n.

Now note that in the new coordinates implied by (1), $\mathbf{N}(\underset{\sim}{E}) = 0 \oplus R^{n-r}$, whence $\mathbf{AN}(E) = \mathbf{R}(\tilde{A})$. Since $\mathbf{R}(E) = \mathbf{R}(\bar{E})$, the theorem follows. ●

We can now write:

Theorem 4.3

There exists a feedback for (1.1) which eliminates impulses and arbitrarily assigns the closed-loop eigenvalues on the reachable subspace $\mathbf{R}$ if and only if the system is controllable at infinity.

Proof:

Note that (4.2) is equivalent to (3.4) with $k = 0$. It is easy to show in this simple case that the sets of linearly independent vectors required in Theorem 4.1 exist. ●

The construction of the feedback F is quite simple under the controllability-at-infinity hypothesis.

5. CONCLUSION

We showed that a maximum of $\dim(E\mathbf{V}^*)$ finite eigenvalues can be assigned, with $\dim(E\mathbf{R})$ eigenvalues arbitrary and the remainder equal to the open-loop unreachable eigenvalues. A freely selected eigenvalue λ_i has its eigenvector in the m-dimensional subspace $(\lambda_i E-A)^{-1}B$, which is contained in the reachable subspace $\mathbf{R}$.

It was shown how to choose the required feedback gain and how to extend it to R^n so as to guarantee regularity and minimize the degree of closed-loop impulsive behavior. If the system is controllable at infinity, impulsive behavior can be eliminated.

All of our constructions were in terms of the original system matrices E,A,B. The computation of $\mathbf{V}^*$, $\mathbf{R}$, and the feedback gain does not require the Weierstrass form, but depends rather on subspace recursions.

REFERENCES

[1] B.C. Moore, "On the Flexibility Offered by State Feedback in Multivariable Systems Beyond Closed Loop Eigenvalue Assignment," IEEE Trans. Automat. Control, pp. 689-692, Oct. 1976.

[2] G. Klein and B.C. Moore, "Eigenvalue-Generalized Eigenvector Assignment with State Feedback," IEEE Trans. Automat. Control, pp. 140-141, Feb. 1977.

[3] M.M. Fahmy and J. O'Reilly, "Eigenstructure Assignment in Linear Multivariable Systems-A Parametric Solution," IEEE Trans. Automat. Control, Vol. AC-28, No. 10, pp. 990-994, Oct. 1983.

[4] G. Klein, "On the Relationships Between Controllability Indexes, Eigenvector Assignment, and Deadbeat Control," IEEE Trans. Automat. Control, Vol. AC-29, No. 1, pp. 77-94, Jan. 1984.

[5] M-I.J. Chang, "Eigenstructure Assignment by State Feedback," Proc. ACC, San Diego, CA, pp. 372-377, June 1984.

[6] E.L. Yip and R.F. Sincovec, "Solvability, Controllability, and Observability of Continuous Descriptor Systems," IEEE Trans. Automat. Control, Vol. AC-26, No. 3, June 1981.

[7] G.C. Verghese, B.C. Lévy, and T. Kailath, "A Generalized State-Space for Singular Systems," IEEE Trans. Automat. Control, Vol. AC-26, No. 4, pp. 811-831, Aug. 1981.

[8] L.R. Fletcher, J. Kautsky, and N.K. Nichols, "Algorithms for Pole Assignment," Proc. IEEE Conf. Decision and Control, Las Vegas, NV, Dec. 1984.

[9] K.T. Wong, "The Eigenvalue Problem $\lambda Tx + Sx$," J. Diff. Eq., Vol. 16, pp. 270-280, 1974.

[10] D. Cobb, "Feedback and Pole Placement in Descriptor Variable Systems," Int. J. Control, Vol. 33, No. 6, pp. 1135-1146, 1981.

[11] K. Ozcaldiran, Control of Descriptor Systems, Ph.D. Thesis, School of Electrical Engineering, Georgia Institute of Technology, Atlanta, GA, June 1985.

[12] G.C. Verghese, "Further Notes on Singular Descriptions," Proc. JACC, Charlottesville, VA, paper TA-4B, June 1981.

[13] V.A. Armentano, "Eigenvalue Placement for Generalized Linear Systems," Sys. and Control Letters, Vol. 4, pp. 199-202, 1984.

[14] W.M. Wonham, Linear Multivariable Control: A Geometric Approach, New York: Springer-Verlag, 1979.

[15] T. Kailath, Linear Systems, New York: Prentice-Hall, 1980.

[16] G.C. Verghese, Infinite-Frequency Behavior in Generalized Dynamical Systems, Ph.D. Thesis, Dept. of Electrical Engineering, Stanford Univ., Stanford, CA 1978.